KTH, Department of Mathematics — 1 June 2020
Examiner: Lilian Matthiesen — 8:00-11:00

Number Theory (SF2728, MM8012) – Final Exam

The preliminary lower bounds for each of the passing grades and Fx are: A:26, B:23, C:20, D:17, E:13, Fx:12.

No advanced calculators or computer algebra systems may be used. Theorems from the course may be used without proof. Any results beyond the course have to be proved if used. Best of luck!

Part A

1. Which of the following complex numbers are algebraic integers. Justify your answers.
   (a) $(\sqrt{23} + i5^{3/7})^{1/5}$ [1p]
   (b) $e^{2\pi i/7} + \frac{1}{2}\sqrt{15}$ [1p]

2. Let K be a number field, let I be a fractional $\mathscr{O}_K$-ideal, and suppose that $1 \in I$. Show that $I = J^{-1}$ for some integral $\mathscr{O}_K$-ideal J. [2p]

3. Let K be a number field of degree n over $\mathbb{Q}$, suppose that $\alpha \in \mathscr{O}_K$, $\alpha \neq 0$, is a non-zero integer, and let $\alpha^{(1)}, \ldots, \alpha^{(n)}$ denote its conjugates with respect to K.
   (a) Show that if $|\alpha^{(1)}|, \ldots, |\alpha^{(n-1)}| < 1$, then α is algebraic of degree n over $\mathbb{Q}$. [2p]
   (b) Does the same conclusion hold if $|\alpha^{(1)}|, \ldots, |\alpha^{(n-2)}| < 1$ and if $\alpha^{(n-1)}, \alpha^{(n)} \in \mathbb{C} \setminus \mathbb{R}$ form a pair of complex conjugates? [1p]

4. Let K be a number field. Show that the group of units of K is finite if and only if $K = \mathbb{Q}$ of $K = \mathbb{Q}(\sqrt{D})$ for some square-free integer $D < 0$. [2p]

5. Let K be a number field and let d_K denote its discriminant. Show that
$$|d_K| \geqslant \left(\frac{\pi}{4}\right)^n \left(\frac{n^n}{n!}\right)^2.$$ [2p]

6. Let K be a number field and $J \subset \mathscr{O}_K$ an integral ideal. Show that $J \mid (\mathrm{N}(J))$. [2p]

Part B

7. Let $D \in \mathbb{N}$ be a natural number. Show that there exists a finite set $S \subset \mathbb{A}$ of algebraic integers such that every number field K with $d_K = D$ is of the form $K = \mathbb{Q}(\alpha)$ for some $\alpha \in S$. [You may use any of the questions from part A.] [6p]

8. Let $K = \mathbb{Q}(\sqrt{-23})$.
   (a) Determine $\mathscr{O}_K$. If $\mathscr{O}_K = \mathbb{Z}[\alpha]$, find the minimal polynomial for α. [1p]
   (b) Determine $Cl(K)$. [5p]

9. Let $q > 2$ be a natural number and let $\chi : \mathbb{N} \to \mathbb{C}$, $\chi \neq \chi_0$, be a non-trivial Dirichlet character modulo q. Show that

$$\sum_{n=1}^{\infty} \frac{\chi(n) \log n}{n}$$

converges. [6p]