KTH, Department of Mathematics
Examiner: Lilian Matthiesen

14 August 2020
14:00-17:00

Number Theory (SF2728, MM8012) – Re-Exam

The preliminary lower bounds for each of the passing grades and Fx are:
A:18, B:16, C:14, D:12, E:10, Fx:9.
No advanced calculators or computer algebra systems may be used. Theorems from the course may be used without proof. Any results beyond the course have to be proved if used. Best of luck!

1. Let $\mathscr{O}_K$ be the ring of integers of a number field K and suppose that there exists a map $n : \mathscr{O}_K \to \mathbb{N}$ with the property that for all $\alpha, \beta \in \mathscr{O}_K$ there exist $\theta, \varrho \in \mathscr{O}_K$ such that $\alpha = \beta\theta + \varrho$ and $n(\varrho) < n(\beta)$ or $\varrho = 0$. Show that every ideal in $\mathscr{O}_K$ is principal. [4p]

2. Let $\delta > 0$ and suppose that $\{a_n\}_{n\in\mathbb{N}}$ is a sequence of complex numbers with the property that
$$A(x) := \sum_{1\leqslant n\leqslant x} a_n \ll x^\delta$$
for all $x > 1$. Show that the Dirichlet series $\alpha(s) = \sum_{n\geqslant 1} a_n n^{-s}$ converges at every $s = \sigma + it$ with $\sigma > \delta$. [4p]

3. By working with ideals and their factorisation in a suitable number field, determine the set of all rational primes $p > 3$ for which there is a solution to the equation $X^2 + 3Y^2 = p$ with $X, Y \in \frac{1}{2}\mathbb{Z} = \{\frac{a}{2} \mid a \in \mathbb{Z}\}$. [6p]

4. Let $D \in \mathbb{Z}_{<0}$, $D \not\equiv 1 \pmod 4$, be a negative square-free number and consider the quadratic field $K = \mathbb{Q}(\sqrt{D})$.
   (a) Determine the discriminant and an integral basis for K. [1p]
   (b) Describe the set of all prime ideals $J \subset \mathscr{O}_K$ such that J^2 is principal but J is not a principal ideal. [1p]
   (c) Determine the number of classes $\mathscr{C} \in Cl(K)$ in the class group of K for which $\mathscr{C}^2 = 1$. [4p]
   [Observe that $(\sqrt{D}) \subset \mathscr{O}_K$. What does this mean for the ideals from (b)? Can you show that this is the only additional relation between these ideals?]

5. Let $c > 0$ be a positive real number, let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of complex numbers, and suppose that the Dirichlet series $\alpha(s) = \sum_{n\geqslant 1} a_n n^{-s}$ is absolutely convergent at $s = c$. Show that for all $x \in \mathbb{R} \setminus \mathbb{Z}$, $x > 0$, we have
$$\sum_{n\leqslant x} \left(1 - \frac{n}{x}\right) a_n = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \alpha(s) \frac{x^s}{s(s+1)} ds.$$
[4p]