KTH, Department of Mathematics — 15 August 2018
Examinator: Lilian Matthiesen — 08:00-11:00

Number Theory (SF2728, MM8012) – Re-Exam

To get grade E (or higher) one needs at least 8p (out of 16p) on the part called Proofs and 12p (out of 24p) on the part called Problems. For higher grades the intervals are: A 40-34p, B 33-30p, C 29-26p, D 25-23p, E 22-20p.

No calculators may be used. Best of luck!

Proofs

1. Let $\theta \in \mathbb{C}$ be algebraic of degree n over $\mathbb{Q}$. Show that every element α of the number field $\mathbb{Q}(\theta)$ has a unique representation as
$$\alpha = x_0 + x_1\theta + \cdots + x_{n-1}\theta^{n-1},$$
where $x_0, \ldots, x_{n-1} \in \mathbb{Q}$. [8p]

2. Let K be a number field, let $\mathcal{O}_K$ denote its ring of integers and let $J_1, J_2 \subset \mathcal{O}_K$ be ideals.
   (a) Suppose that $J_2 = (\gamma)$ is principal. Show that $J_1 \subset J_2 \implies J_1 = J_2 J$ for some ideal $J \subset \mathcal{O}_K$. [2p]
   (b) Show that $J_1 \subset J_2$ if and only if $J_2 | J_1$. [6p]

Problems

3. Let $\alpha \in \mathbb{C}$ be an algebraic number. Show that there exists a rational integer r such that αr is an algebraic integer. [5p]

4. Let K be a number field and suppose that $\beta_1, \ldots, \beta_n \in \mathcal{O}_K$ are linearly independent over $\mathbb{Q}$. Show that if $\mathrm{disc}_K(\beta_1, \ldots, \beta_n)$ is square-free, then the n numbers $\beta_1, \ldots, \beta_n$ form a $\mathbb{Z}$-basis for $\mathcal{O}_K$. [4p]

5. Let p be an odd prime and let $\left(\frac{\cdot}{p}\right)$ denote the Legendre symbol.
   (a) Compute
   $$\sum_{n=m}^{m+p-1} \left(\frac{n}{p}\right)$$
   for any $m \in \mathbb{Z}$. [4p]
   (b) State a suitable version of the partial summation identity and show that
   $$\sum_{n=1}^{\infty} \left(\frac{n}{p}\right) n^{-s}$$
   is convergent for all positive $s \in \mathbb{R}$. [4p+4p]

   [You may use the fact that there are exactly $\frac{p-1}{2}$ quadratic residues modulo p.]

6. Using the Minkowski bound or otherwise compute the class number of $\mathbb{Q}(\sqrt{13})$. (Minkowski's constant is given by $\frac{n!}{n^n}(4/\pi)^s$.) [3p]