

NUMBER THEORY (SF2728, MM8012) – FINAL EXAM

To get grade E (or higher) one needs at least 10p (out of 20p) on the part called Proofs and 17p (out of 34p) on the part called Problems. For higher grades the intervals are: A 54-46p, B 45-41p, C 40-36p, D 35-31p, E 30-27p.

No calculators may be used. Best of luck!

PROOFS

1. (a) Let  $K$  be a number field and let  $I_1, I_2, J \subseteq \mathcal{O}_K$  be integral ideals and  $J \neq (0)$ . Show that if  $I_1J = I_2J$ , then  $I_1 = I_2$ . [5p]
- (b) Let  $K$  be a number field. Prove that every prime ideal  $J \subseteq \mathcal{O}_K$  divides precisely one ideal  $p\mathcal{O}_K$ , where  $p$  is a rational prime. [5p]

2. Let  $x, y \in \mathbb{N}$  be integers and suppose that  $x > y > 0$ . Let  $(a_n)_{n \in \mathbb{N}}$  be a sequence of complex numbers and define  $A_x = \sum_{y \leq n \leq x} a_n$ . Let  $g : [y, x] \rightarrow \mathbb{C}$  be continuously differentiable.  
(a) Show that

$$\sum_{y \leq n \leq x} a_n g(n) = A(x)g(x) - \int_y^x A(t)g'(t)dt.$$

[5p]

- (b) Assuming the prime number theorem, deduce that

$$\sum_{\substack{p \leq x \\ p \text{ prime}}} \frac{1}{p} \sim \log \log x.$$

[5p]

PROBLEMS

3. Consider the quadratic field  $\mathbb{Q}(\sqrt{D})$ , where  $D$  is square-free and  $D \equiv 2, 3 \pmod{4}$ . Show that  $(1, \sqrt{D})$  is an integral basis. Compute the discriminant of  $\mathbb{Q}(\sqrt{D})$ . [8p]

4. Let  $a, b \in \mathbb{Z}$  and let  $p$  be an odd prime.
- (a) Define the Legendre symbol  $\left(\frac{a}{p}\right)$ . [1p]
  - (b) Show that there are exactly  $\frac{p-1}{2}$  quadratic residues modulo  $p$ . [4p]
  - (c) Show that  $\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right) \left(\frac{b}{p}\right)$ . [4p]
5. Let  $K$  be a number field.
- (a) Define the notion of fractional ideal. [2p]
  - (b) Define the class group of  $K$ . [2p]
  - (c) Using the Minkowski bound or otherwise compute the class number of  $\mathbb{Q}(\sqrt{-7})$ . (Minkowski's constant is given by  $\frac{n!}{n^n}(4/\pi)^s$ .) [4p]
6. (a) Define what it means for  $\alpha \in \mathbb{C}$  to be an algebraic integer. [1p]
- (b) Show that if  $\alpha, \beta \in \mathbb{C}$  are algebraic *numbers*, then  $\alpha + \beta$  is algebraic. [5p]
  - (c) Determine the units of  $\mathbb{Q}(\sqrt{-1})$ . [3p]