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Number Theory
Final Exam

Friday, June 3, 2016

Instructions: Closed book, no notes and no calculators allowed. Unless indicated otherwise, you may quote facts that appear in the textbook, or were proved in class. When you do, state precisely any result you are using. Be sure to justify your answers, and show clearly all steps of your solutions. Results of previous problems can be used even if you could not solve them

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Part 1: Proofs

1 [10 points] Prove that $\Sigma \frac{1}{p} = \infty$, where p ranges over the set of (positive) primes.

2 [10 points] Let p be an odd prime. Prove that there exist integers a, b such that $p = a^2 + b^2$ if and only if $p \equiv 1 \pmod 4$.

Note: You may (if you wish) use Gauss or Jacobi sums and any properties of those that were proved in class or in the textbook. If you do, give a definition and state clearly what properties you are using.

3 [10 points] Let $\zeta(s)$ be the Riemann zeta function. Prove that the function $\zeta(s) - \frac{1}{s-1}$ can be extended to an analytic function defined on the domain $\{s \in \mathbb{C} \mid \mathcal{R}e(s) > 0\}$.

Part 2: Problems

4 (a) [5 points] Does the following equation have a solution? (if yes, do not try to find it).

$$2x^2 \equiv -21 \pmod{79}$$

(b) [5 points] How many solutions does the following equation have? Find them

$$x^2 - x - 12 \equiv 0 \pmod{77}$$

5 For parts (a) and (b), let F be an algebraic number field, $\mathcal{O}_F$ the ring of integers of F, and I, J non-zero ideals of $\mathcal{O}_F$

(a) [3 points] Express the prime factorization of $I \cap J$ in terms of the factorizations of I and J.

(b) [1 points] Does $I \cap J$ always equal IJ? Prove, or give a counterexample.

(c) [2 points] Is $\frac{1+\sqrt{-11}}{2}$ an algebraic integer? Justify your answer.

(d) [4 points] Consider the algebraic number field $\mathbb{Q}(\sqrt[3]{2})$. Let $\alpha = 3\sqrt[3]{4}$. Clearly, $\alpha \in \mathbb{Q}(\sqrt[3]{2})$. Find the trace and norm of α.

6 (a) [3 points] How many multiplicative characters modulo 12 are there?

(b) [2 points] Give an explicit formula for one **non-trivial** character modulo 12.

(c) [5 points] Let p be a prime, with $p \equiv 1 \pmod 4$. Prove that the set of primes q such that p is a quadratic residue mod q has Dirichlet density $\frac{1}{2}$. Suggestion: use quadratic reciprocity and Dirichlet's arithmetic progression theorem. Note: the statement remain valid if p is any integer that is not a square, but it may be easier to prove with the additional assumption on p.