MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
Examinator: Jonas Bergström

Tentamensskrivning i
Number Theory, 7,5 hp
15 August 2014
9.00-14.00

To get grade E (or higher) one needs at least 15p (out of 30p) on the part called Proofs and 15p (out of 30p) on the part called Problems.
Higher grades are given by the following intervals: A 60-50p, B 50-45p, C 45-40p, D 40-35p, where one also needs to have at least 18p on the part called Problems for C (respectively 21p for B and 24p for A).

No calculators or computers may be used. Best of luck!

Proofs

In this part you should prove some theorems from the book and you can use all previous results in the book, but do not forget to properly define the main notions and to state any theorem/proposition/lemma that you use.

1. Show that, for any integers a, b and $m > 0$, the congruence $ax \equiv_m b$ has solutions iff $\gcd(a, m)|b$. Show also that if $\gcd(a, m)|b$ then there are precisely $\gcd(a, m)$ solutions. 10 p

2. Let p be a prime, χ a character on $\mathbb{F}_p$ and define the Gauss sum $g(\chi) := \sum_{t\in\mathbb{F}_p} \chi(t)e^{2\pi it/p}$. Prove that if χ is non-trivial then $|g(\chi)| = \sqrt{p}$. 10 p

Lemma 1 *Let F be a number field with norm N. There exists a positive integer M depending only on F with the following property. Given $\alpha, \beta \in D$, $\beta \neq 0$, there is an integer t, $1 \leq t \leq M$ and an element $\omega \in D$ such that $|N(t\alpha - \omega\beta)| < |N(\beta)|$.*

3. Use Lemma 1 above to prove that the class number of F is finite. 10 p

Problems

4. a) Define the Jacobi symbol $\left(\frac{a}{b}\right)$ for any odd positive integer b and any integer a. 1 p

   b) Evaluate the Legendre symbol $\left(\frac{78}{83}\right)$. 2 p

   A prime of the form $p = 2^{2^n} + 1$ for some $n \geq 0$ is called a Fermat prime.

   c) Let $p > 5$ be a Fermat prime. Show that $5^{2^{(2^n-1)}} \equiv_p -1$. 3 p

   d) Show that 5 is a primitive root modulo any Fermat prime $p > 5$. 4 p

5. a) Define what an irreducible element and what a prime element of a domain is. 1 p

   b) Factor the element $13 + \sqrt{-2} \in \mathbb{Z}[\sqrt{-2}]$ into irreducible elements. 2 p

   c) Prove or disprove: The ring $\mathbb{Z}[\sqrt{-3}]$ is a UFD. 3 p

   d) State what it means that a ring has unique factorization of ideals. 1 p

   e) Let A and B be two ideals in a ring of integers D_F of an algebraic number field F. Express the factorization of the ideal $A + B$ in terms of the factorizations of A and B and prove that it holds. 3 p

6. a) Give the definition of the Riemann zeta function $\zeta(s)$ as a sum over all integers (which converges for all $s \in \mathbb{C}$ such that $\Re(s) > 1$). 1 p

b) It is shown in Ireland and Rosen that the equality

$$\zeta(s) = \frac{s}{s-1} - s \int_1^\infty <x> x^{-s-1} dx$$

holds for all $s \in \mathbb{C}$ such that $\Re(s) > 1$ and where for any $x \in \mathbb{R}$, $<x> := x - [x]$ and $[x]$ denotes the greatest integer less than or equal to x. Explain why the right hand side of the equality is well defined for all $s \in \mathbb{C}$ such that $0 < \Re(s) < 1$. 3 p

c) State the Riemann hypothesis for $\zeta(s)$. 1 p

d) Write the Riemann zeta function as a product (it is called an "Euler product") over all primes (which converges for all $s \in \mathbb{C}$ such that $\Re(s) > 1$). 1 p

e) Prove that if $\sigma := \Re(s) > 1$ then $\zeta(s) \neq 0$.
Hint: Use the Euler product description of $\zeta(s)$ and that $1 + y < e^y$ for all real $y \leq 1/2$ to prove that

$$|\zeta(s)| \geq \prod_p e^{-p^{-\sigma}} \geq \exp(-\sum_n n^{-\sigma}),$$

where the product is over all primes p and the sum is over all integers $n \geq 1$. 4 p

The exam will be returned 14.00 on Friday the 29th of August in room 410 in house 6. After that it can be collected in room 204 in house 6.