NON-GAUSSIAN WAVES IN ŠEBA'S BILLIARD

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ABSTRACT. The Šeba billiard, a rectangular torus with a point scatterer, is a popular model to study the transition between integrability and chaos in quantum systems. Whereas such billiards are classically essentially integrable, they may display features such as quantum ergodicity [11] which are usually associated with quantum systems whose classical dynamics is chaotic. Šeba proposed that the eigenfunctions of toral point scatterers should also satisfy Berry's random wave conjecture, which implies that the value distribution of the eigenfunctions ought to be Gaussian. However, Keating, Marklof and Winn formulated a conjecture which suggested that Šeba billiards with irrational aspect ratio violate the random wave conjecture, and we show that this is indeed the case. More precisely, for tori having Diophantine aspect ratio, we construct a subsequence of the set of new eigenfunctions having even/even symmetry, of essentially full density, and show that its fourth moment is not consistent with a Gaussian value distribution. In fact, given any set Λ interlacing with the set of unperturbed eigenvalues, we show non-Gaussian value distribution of the Green's functions G_λ, for λ in an essentially full density subsequence of Λ.

1. Introduction

Šeba's billiard, a rectangular billiard $\mathcal{M}$ with irrational aspect ratio and a Dirac mass placed in its interior, is a popular model in the field of Quantum Chaos to investigate the transition between chaos and integrability in quantum systems. The model was originally proposed by Petr Šeba in 1990 [14] and has since attracted much attention in the literature [6, 4, 5, 2, 3, 10, 13, 17, 11, 18, 9, 12]. Although, the Dirac mass only affects a measure zero subset of trajectories in phase space and thus has essentially no effect on the classical dynamics, Šeba argued that the wave functions of the associated quantized billiard may display similar features as quantum systems which are classically chaotic.

In particular, Šeba conjectured that the wave functions should obey Berry's random wave model, i.e. be well approximated by a superposition of monochromatic random waves as the eigenvalue tends to infinity. Consequently (cf. [1], p. 240, eqs. (78-80)) the moments of the eigenfunctions should converge

Date: September 12, 2021.

P.K. was partially supported by the Swedish Research Council (2016-03701,2020-04036). H.U. was partially supported by the grant ANR-17-CE40-0011-01 of the French National Research Agency ANR (project SpInQS).

to the Gaussian moments in the limit as the eigenvalue tends to infinity (and hence the eigenfunctions should have Gaussian value distribution in the said limit.) In particular, denoting an L^2-normalized (real) wave function with eigenvalue λ by ψ_λ, one expects that the fourth moment of ψ_λ (possibly after excluding a zero density subsequence of exceptional eigenvalues) converge to the corresponding Gaussian moment as $\lambda \to \infty$, namely that

$$\mathbb{E}(\psi_\lambda^4) = \int_{\mathcal{M}} \psi_\lambda^4 d\mu \to 3,$$

where $d\mu = d\mu_{Leb.} / \operatorname{vol}(\mathcal{M})$ denotes the normalized Lebesgue measure.

Šeba calculated the value distribution for high energy wave functions and found seemingly strong numerical evidence for a Gaussian value distribution in line with Berry's predictions. Later Keating, Marklof and Winn cast doubt on Šeba's conjecture when they showed that quantum star graphs, a model believed to be similar in behaviour to Šeba's billiard, did indeed violate the random wave model [10, 3].

In this paper we put this matter to rest by showing that for a Šeba billiard with Diophantine aspect ratio (a condition that holds generically), the fourth moment of the eigenfunctions cannot tend to a Gaussian. In fact we can find a subsequence of arbitrarily high density such that the fourth moment stays strictly (and uniformly once the density is fixed), *below* the Gaussian fourth moment as the eigenvalue tends to infinity, which in particular rules out a Gaussian value distribution. In fact, our results are valid for any sequence of numbers which interlace with the Laplace eigenvalues, in particular for the new eigenvalues of the both the weak and the strong coupling quantizations of the Šeba billiard. The former arises from von Neumann's theory of self-adjoint extensions, whereas the latter, investigated numerically in Šeba's paper, uses a different renormalization which is considered more physically relevant (cf. [16] for a detailed discussion of weak and strong coupling quantizations.)

1.1. **Background.** Before we state the results, let us recall the mathematical definition of Šeba's billiard. In this paper we will mainly focus on periodic boundary conditions (the case of Dirichlet boundary conditions is treated in the Appendix) and thus deal with a flat 2-torus $\mathbb{T}^2 = \mathbb{R}^2/2\pi\mathcal{L}_0$, where $\mathcal{L}_0 = \mathbb{Z}(a, 0) \oplus \mathbb{Z}(0, 1/a)$ for some $a > 0$ such that a^4 is a *Diophantine* number (cf. [7, Ch. 2.8]). The formal Schrödinger operator associated with a Dirac mass placed at the point $x_0 \in \mathbb{T}^2$ is given by

$$-\Delta + \alpha\delta_{x_0}.$$

This formal operator may be associated with a one-parameter family of self-adjoint extensions of the restricted positive Laplacian $-\Delta|_{C_c^\infty(\mathbb{T}^2\setminus\{x_0\})}$. For the details of this theory we refer the reader to the introduction and appendix of the paper [13]. We adopt the notation of this paper and refer to the self-adjoint extensions as $-\Delta_\varphi$, where $\varphi \in (-\pi, \pi)$ is the extension parameter.

One of the key features of the spectral theory of the operator $-\Delta_\varphi$ is that it represents a rank-one perturbation of the Laplacian. That is, for each Laplace eigenspace the perturbation "tears off" a new eigenvalue, and the spectrum of $-\Delta_\varphi$ therefore consists of two parts: the "old" and the "new" eigenvalues. The multiplicity of each old eigenvalue is reduced by one and the corresponding eigenspace is just the co-dimension one subspace of Laplace eigenfunctions which vanish at x_0. This part of the spectrum is therefore not affected by the presence of the Dirac mass. On the other hand, the new part of the spectrum "feels" the presence of the scatterer and the 4th moment of these "new eigenfunctions" will be the focus of this paper.

The new eigenvalues interlace with the old Laplace eigenvalues and the associated eigenfunctions are just Green's functions which, on letting $\mathcal{L}$ denote the dual lattice of $\mathcal{L}_0$, have the following L^2-expansion:

$$G_\lambda(x, x_0) = \sum_{\xi \in \mathcal{L}} \frac{e^{\mathrm{i}\langle \xi, x - x_0\rangle}}{|\xi|^2 - \lambda}, \tag{1.1}$$

with the following formula for the 2nd moment:

$$\int_{\mathbb{T}^2} |G_\lambda(x, x_0)|^2 d\mu(x) = \sum_{\xi \in \mathcal{L}} \frac{1}{(|\xi|^2 - \lambda)^2},$$

where $d\mu(x) = dx/(4\pi^2)$ denotes the normalized Lebesgue measure on $\mathbb{T}^2$.

The set of new eigenvalues can be determined as the solutions of a spectral equation [13]. There is in fact another quantization condition — known as a strong coupling quantization — which is considered more relevant in the physics literature and requires a renormalization of the self-adjoint extension parameter φ as the eigenvalue λ increases. This leads to a different spectral equation, but as our results will in fact hold for any sequence which interlaces with the unperturbed eigenvalues we will not dwell on this matter (details can be found in [16, 15].)

1.2. **Results.** Let us denote by $g_\lambda = G_\lambda / \|G_\lambda\|_2$ the L^2-normalized new eigenfunctions. The following theorem is our main result and shows that the fourth moment of eigenfunctions of Šeba's billiard is not Gaussian, in particular that the value distribution of the wave functions is not consistent with a Gaussian distribution in the limit as the eigenvalue λ tends to infinity — a contradiction to Berry's random wave model. (Note that a Gaussian value distribution implies that the even moments of eigenfunctions are bounded *below* by the corresponding even moments of the Gaussian, provided the variances are normalized to be one. To see this for the $2k$-th moment, take continuous bounded minorants of x^{2k}, e.g. $f_{2k,t}(x) := \min(x^{2k}, t^{2k})$, for a sequence of t's tending to infinity.)

Theorem 1.1. *Consider a 2-torus with* Diophantine *aspect ratio, and let $\Lambda \subset \mathbb{R}$ denote any subset interlacing with the set of unperturbed eigenvalues. Given $\epsilon \in (0, 1)$ there exists a subsequence of Λ, of relative density $1 - \epsilon$,* *and a constant $C_\epsilon > 0$ such that for λ tending to infinity along the said subsequence we have*

$$1 - o(1) \leq \mathbb{E}(g_\lambda^4) \leq 3 - C_\epsilon + o(1).$$

Remark. *As the torus is homogenous we may place the scatterer at $x_0 = 0$ and it is then natural to desymmetrize with respect to odd/even-ness vis-a-vis horizontal and vertical reflections, and the set of new eigenfunctions then corresponds exactly to eigenspaces having even/even-invariance. (To see this, note that the three eigenfunctions of the unperturbed Laplacian having odd/odd, odd/even, or even/odd symmetry, all vanish at 0 and are thus also "old" eigenfunctions of the perturbed Laplacian, whereas the "new" eigenfunction is given by an* even *Green's function; to see this take $x_0 = 0$ in (1.1) and note that $\mathcal{L}$ is invariant under both reflections.) Thus, within the even/even symmetry class, essentially all of the eigenfunctions have non-Gaussian fourth moments.*

1.3. **Acknowledgements.** We would like to thank Zeev Rudnick for very helpful discussions and comments on the manuscript. We would also like to thank Stéphane Nonnenmacher for reading the manuscript very carefully and giving many detailed comments improving the exposition as well as the result — in particular for raising the question whether the results hold for any sequence interlacing with the unperturbed eigenvalues. Finally, we are very grateful to the anonymous referee for many comments greatly improving the exposition.

2. Approximating the 4th moment

We begin with the following notational convention: we denote by $f \ll g$ that there exists a constant $C > 0$ s.t. $|f(x)| \leq Cg(x)$ for all sufficiently large x.

2.1. L^4 **convergence.** Let $\mathcal{L}_0$ be a *Diophantine* irrational rectangular unimodular lattice, as defined above, and consider the 2-torus $\mathbb{T}^2 = \mathbb{R}^2/2\pi\mathcal{L}_0$. Fix $\lambda > 0$ a new eigenvalue, and for $\xi \in \mathcal{L}$ define $c_\lambda(\xi) := (|\xi|^2 - \lambda)^{-1}$.

The following expansion for the Green's function holds in the L^2-sense:

$$G_\lambda(x) := G_\lambda(x, 0) = \sum_{\xi \in \mathcal{L}} c_\lambda(\xi) e^{i\xi\cdot x} \tag{2.1}$$

(without loss of generality we may assume that $x_0 = 0$.) Our aim is, first of all, to show that this expansion also holds in the L^4-sense. We thus introduce the truncated Green's function

$$G_\lambda^T(x) = \sum_{\xi \in \mathcal{L}, |\xi| \leq T} c_\lambda(\xi) e^{i\xi\cdot x}, \quad T \geq 10\lambda^{1/2},$$

and show that G_λ^T converges in $L^4(\mathbb{T}^2)$, as $T \to \infty$.

We will achieve this by showing that G_λ^T is Cauchy in $L^4(\mathbb{T}^2)$, in particular we will bound the L^4-norm of the difference $G_\lambda^{2T} - G_\lambda^T$. Letting

$$A(T) := \{v \in \mathcal{L} : |v| \in [T, 2T]\}$$

we find that (recall $T \geq 10\lambda^{1/2}$, and thus $c_\lambda(v) > 0$ for $v \in A(T)$)

(2.2)
$$\begin{aligned}\int_{\mathbb{T}^2} |G_\lambda^{2T}(x) - G_\lambda^T(x)|^4 d\mu(x) &= \sum_{v_1,v_2,v_3,v_4 \in A(T) : \sum_{i=1}^4 v_i = 0} \prod_{i=1}^4 c_\lambda(v_i) \\ &\ll \sum_{v_1,v_2,v_3,v_4 \in A(T) : \sum_{i=1}^4 v_i = 0} \frac{1}{|v_1|^2|v_2|^2|v_3|^2|v_4|^2} \\ &\ll \frac{1}{T^8} \cdot |\{v_1, v_2, v_3, v_4 \in A(T) : \sum_{i=1}^4 v_i = 0\}|\end{aligned}$$

and, since $v_4 = -\sum_{i=1}^3 v_i$, we find that the number of 4-tuples is at most $|A(T)|^3 \ll (T^2)^3$, and thus the above is $\ll \frac{1}{T^2}$. Hence, for any $T \geq 10\lambda^{1/2}$ we have

$$\|G_\lambda^{2T} - G_\lambda^T\|_4 \ll T^{-1/2}.$$

Thus, for $T \geq 10\lambda^{1/2}$ and any integer $k \geq 0$,

$$\|G_\lambda^{2^{k+1}T} - G_\lambda^{2^k T}\|_4 \ll 2^{-k/2} T^{-1/2}$$

which implies that for any integers $p > q > 0$

$$\|G_\lambda^{2^p T} - G_\lambda^{2^q T}\|_4 \ll T^{-1/2} \sum_{k=q}^{p-1} 2^{-k/2} \ll T^{-1/2} 2^{-q/2}.$$

Hence, by telescopic summation, we find that $(G_\lambda^{2^q T})_q$ is a Cauchy sequence and therefore converges to a limit in L^4 as $q \to \infty$. An argument similar to the one used above shows that if $\tilde{T} \in [2^k T, 2^{k+1} T]$ then $\|G_\lambda^{2^k T} - G_\lambda^{\tilde{T}}\|_4 \ll T^{-1/2} 2^{-k/2}$, and thus $(G_\lambda^{T_i})_{i \geq 1}$ is also a Cauchy sequence for any countable sequence $T_1 < T_2 < T_3 \ldots$ tending to infinity.

In particular, we have

$$\|G_\lambda\|_4^4 = \sum_{v_1,v_2,v_3 \in \mathcal{L}} c_\lambda(v_1) c_\lambda(v_2) c_\lambda(v_3) c_\lambda(v_1 + v_2 - v_3). \tag{2.3}$$

2.2. **Further truncations.** Let $L = L(\lambda)$ be an increasing function such that $L \to +\infty$ as $\lambda \to +\infty$, and let $A(\lambda, L)$ denote the annulus

$$A(\lambda, L) := \{v \in \mathcal{L} : |v|^2 \in [\lambda - L, \lambda + L]\}$$

We introduce the Green's function truncated to lattice points inside the annulus $A(\lambda, L)$

$$G_{\lambda,L}(x) = \sum_{\xi \in A(\lambda,L)} c_\lambda(\xi) e^{i\xi \cdot x}, \quad c_\lambda(\xi) := \frac{1}{|\xi|^2 - \lambda}. \tag{2.4}$$

We have the following lemma which shows that $G_{\lambda,L}$ approximates G_λ in $L^4(\mathbb{T}^2)$ as $\lambda \to \infty$ if $L = L(\lambda)$ is an increasing function of λ tending to infinity, as long as a mild growth condition is imposed.

Lemma 2.1. *Let $L = L(\lambda)$ be an increasing function that tends to infinity with λ in such a way that $10 \leq L(\lambda) \leq \lambda/2$ for all $\lambda > 0$. There exists a full density subsequence of new eigenvalues such that, for all λ in the said subsequence, we have*

$$\|G_\lambda - G_{\lambda,L}\|_4 \ll L^{-1/4+o(1)}.$$

Proof. Let $A_+ = A_+(\lambda, L)$ denote the set $\{v \in \mathcal{L} : |v|^2 > \lambda + L\}$ and by $A_- = A_-(\lambda, L)$ the disk $\{v \in \mathcal{L} : |v|^2 < \lambda - L\}$. We begin by noting that

$$\|G_\lambda - G_{\lambda,L}\|_4^4 = \sum_{v_1,\dots v_4 \in A_+ \cup A_- : \sum_{i=1}^4 v_i = 0} \prod_{i=1}^4 \frac{1}{|v_i|^2 - \lambda};$$

writing $G_\lambda - G_{\lambda,L} = \sum_{v \in A_+} + \sum_{v \in A_-}$ and using the L^4 triangle inequality we can treat large and small v separately. We begin by showing that

$$\sum_{v_1,\dots v_4 \in A_+(\lambda,L) : \sum_{i=1}^4 v_i = 0} \prod_{i=1}^4 \frac{1}{|v_i|^2 - \lambda}$$

is small (for most $\lambda \in \Lambda$) given that L tends to infinity as λ grows. Up to a bounded combinatorial factor, we may after reordering terms assume that $|v_{i+1}|^2 - \lambda \geq |v_i|^2 - \lambda > 0$ for $i = 1, 2, 3$, hence $|v_4|^2 - \lambda \geq \prod_{i=1}^3 (|v_i|^2 - \lambda)^{1/3}$; on noting that v_4 is determined by v_1, v_2, v_3, it is enough to show that

$$\prod_{i=1}^3 \left(\sum_{|v_i|^2 \geq \lambda + L} \frac{1}{(|v_i|^2 - \lambda)^{4/3}} \right) \ll L^{-1+o(1)}$$

In particular, it is enough to show that $\sum_{|v|^2 \geq \lambda + L} \frac{1}{(|v|^2-\lambda)^{4/3}} \ll L^{-1/3+o(1)}$; which in turn reduces to showing that

$$\sum_{2\lambda \geq |v|^2 \geq \lambda + L} \frac{1}{(|v|^2 - \lambda)^{4/3}} \ll L^{-1/3+o(1)}$$

(to see this, use Weyl's law and partial summation to bound the contribution from v such that $|v|^2 > 2\lambda$.)

Now, given an integer $k \geq 0$, let $M(k)$ denote the number of unperturbed eigenvalues in the interval $[k, k+1)$, or equivalently, the number of lattice points v such that $|v|^2 \in [k, k+1)$. We consider the sum over all $\lambda \in (T/2, T) \cap \Lambda$, and show that dyadic means of $\sum_{2\lambda \geq |v|^2 \geq \lambda + L} \frac{1}{(|v|^2-\lambda)^{4/3}}$ are small for T large. More precisely,

$$\sum_{\lambda \in \Lambda \cap (T/2, T)} \sum_{2\lambda \geq |v|^2 \geq \lambda + L} \frac{1}{(|v|^2 - \lambda)^{4/3}} \ll \sum_{l < T} M(l) \sum_{2T \geq k \geq L} \frac{M(l+k)}{k^{4/3}}$$

which, using the same argument as in the proof of [12, Lemma 3.3] (here it is *crucial* that a^4 is Diophantine) is

$$\ll \sum_{2T\geq k\geq L} \frac{1}{k^{4/3}} \sum_{l<T} M(l)M(k+l) \ll \sum_{2T\geq k\geq L} \frac{1}{k^{4/3}} \sum_{l<T} M(l)^2 \ll L^{-1/3}T.$$

Hence, using Chebychev's inequality, for most $\lambda \in \Lambda \cap (T/2, T)$ we find that

$$\sum_{v\in\mathcal{L}:|v|^2\geq\lambda+L} \frac{1}{(|v|^2-\lambda)^{4/3}} \ll L^{-1/3+o(1)}. \tag{2.5}$$

A similar argument shows that, for most $\lambda \in \Lambda \cap (T/2, T)$,

$$\sum_{v\in\mathcal{L}:|v|^2\leq\lambda-L} \frac{1}{(|v|^2-\lambda)^{4/3}} \ll L^{-1/3+o(1)} \tag{2.6}$$

and hence the L^4 norm is $\ll L^{-1/4+o(1)}$.

□

3. Proof of Theorem 1.1

One finds (cf. the L^2-expansion of the Green's function (2.1), or see [13, eq. (3.22)]) that

$$\|G_\lambda\|_2^2 = \int_{\mathbb{T}^2} |G_\lambda|^2 d\mu = \sum_{\xi\in\mathcal{L}} \frac{1}{(|\xi|^2-\lambda)^2} = \sum_{n\in\mathcal{N}} \frac{r_{\mathcal{L}}(n)}{(n-\lambda)^2}, \tag{3.1}$$

where $r_{\mathcal{L}}(n)$ is the multiplicity of the Laplace eigenvalue n and

$$\mathcal{N} = \{n_0 = 0 < n_1 < n_2 < \cdots\}$$

denotes the set of distinct (unperturbed) Laplace eigenvalues.

Also (cf. (2.3)),

$$\begin{aligned} &\|G_\lambda\|_4^4 = \int_{\mathbb{T}^2} |G_\lambda|^4 d\mu = \\ &\sum_{\substack{\xi_1+\xi_2=\eta_1+\eta_2 \\ \xi_1,\xi_2,\eta_1,\eta_2\in\mathcal{L}}} \frac{1}{(|\xi_1|^2-\lambda)(|\xi_2|^2-\lambda)(|\eta_1|^2-\lambda)(|\eta_2|^2-\lambda)}. \end{aligned} \tag{3.2}$$

3.1. **The sequence Λ_g.** We recall some useful results from sections 6 and 7 of [13]. Let $\theta < 1/3$ denote the best known exponent in the error term for the circle problem for a rectangular lattice [8]. In fact, we will only need $\theta < 1/2$, just a bit beyond the trivial geometric estimate. Adopting the notation of [13] we let $\delta \in (0, \frac{2}{3}(\frac{1}{2}-\theta))$ and define

$$S(\lambda) = \bigcup_{\substack{0\neq\zeta\in\mathcal{L} \\ |\zeta|<\lambda^{\delta/2}}} S_\zeta.$$

where we define S_ζ for any $\zeta \in \mathcal{L} \setminus \{0\}$ as the set of solutions to a certain diophantine inequality (cf. eq. (6.1) in [13]), namely

$$S_\zeta := \{\eta \in \mathcal{L} \mid |\langle \eta, \zeta \rangle| \leq |\eta|^{2\delta}\}.$$

We will show that the subset of "good" eigenvalues

$$\Lambda_g := \{\lambda \in \Lambda \mid A(\lambda, \lambda^\delta) \cap S(\lambda) = \emptyset\}$$

is of full density in Λ (recall that $|\{\lambda \in \Lambda : \lambda \leq X\}| \sim X$), by showing that

$$\{\lambda \in \Lambda \setminus \Lambda_g \mid \lambda \leq X\} \ll X^{1-\delta_0}$$

for $\delta_0 = \frac{1}{2} - \theta - \frac{3}{2}\delta > 0$. To see this, denote the complement of Λ_g, i.e. the set of "bad" elements, by $\Lambda_b := \Lambda \setminus \Lambda_g$.

We then have the inclusion

$$\{\lambda \in \Lambda_b \mid \lambda \leq X\} \subset \bigcup_{\substack{0 \neq \zeta \in \mathcal{L} \\ |\zeta| < X^{\delta/2}}} B_\zeta \tag{3.3}$$

where $B_\zeta = \{\lambda \in \Lambda \mid A(\lambda, \lambda^\delta) \cap S_\zeta \neq \emptyset\}$. (To see this, note that $\lambda \in \Lambda \setminus \Lambda_g$ and $\lambda \leq X$ implies that $A(\lambda, \lambda^{\delta/2}) \cap S_\zeta \neq \emptyset$ for some nonzero ζ with $|\zeta| < \lambda^{\delta/2} \leq X^{\delta/2}$, and hence $\lambda \in B_\zeta$ for some nonzero $\zeta \in \mathcal{L}$ with $|\zeta| \leq X^{\delta/2}$.)

We next recall the bound (6.4) in [13], namely that, for fixed $\zeta \in \mathcal{L}$,

$$|\{\lambda \in B_\zeta \mid \lambda \leq X\}| \leq \frac{X^{1/2+\theta+\delta}}{|\zeta|}.$$

(Note that in the proof of (6.4), the only property used regarding the location of the λ's is the interlacing property. Further, the lower bound $\theta/2 > \delta$, stated at the beginning of [13, Section 6], is not used in order to prove (6.4).) We may now apply this bound to get an estimate on the number of bad eigenvalues $\lambda \leq X$. Note that we are summing over lattice vectors $\zeta \in \mathcal{L}$ which are not too large, i.e. $|\zeta| < \lambda^{\delta/2} \leq X^{\delta/2}$, and we find that

$$|\{\lambda \in \Lambda_b \mid \lambda \leq X\}| \leq X^{1/2+\theta+\delta} \sum_{\substack{0 \neq \zeta \in \mathcal{L} \\ |\zeta| < X^{\delta/2}}} \frac{1}{|\zeta|} \ll X^{1/2+\theta+3\delta/2} = X^{1-\delta_0}$$

where $\delta_0 = \frac{1}{2} - \theta - \frac{3}{2}\delta > 0$ (we stress that only the condition $0 < \delta < \frac{2}{3}(\frac{1}{2} - \theta)$ is required).

3.2. **Diagonal solutions.** We begin with the following Lemma which shows that if $\lambda \in \Lambda_g$, then $A(\lambda, L)$ contains only lattice points that are reasonably well-spaced. Recall that a^2 is the aspect ratio of the lattice $\mathcal{L}$.

Lemma 3.1. *Let $1 \leq \lambda \in \Lambda_g$ and put $L = L(\lambda) := \frac{1}{20}\min(a, 1/a)\lambda^{\delta/2}$. If ξ and η are two distinct lattice points belonging to $A(\lambda, L)$, then $|\xi - \eta| \geq \lambda^{\delta/2}$.*

Proof. To see this, put $\beta = \eta - \xi$ and suppose for contradiction that $|\beta| = |\eta - \xi| < \lambda^{\delta/2}$. As $\lambda \in \Lambda_g$, and $\xi \in A(\lambda, L)$ we find that

$$||\xi|^2 - \lambda| = ||\eta - \beta|^2 - \lambda| < L$$

and after multiplying out we obtain

$$\left| |\eta|^2 - \lambda + |\beta|^2 - 2\left\langle \eta, \beta \right\rangle \right| < L.$$

Now, since $|\beta| < \lambda^{\delta/2}$ and $\eta \in A(\lambda, L)$, it follows

$$2|\left\langle \eta, \beta \right\rangle| < \left| |\eta|^2 - \lambda + |\beta|^2 \right| + L \leq \left| |\eta|^2 - \lambda \right| + |\beta|^2 + L < 2L + \lambda^\delta$$

and, since our assumption implies $L < \frac{1}{4}\lambda^{\delta/2}$,

$$|\left\langle \eta, \beta \right\rangle| < L + \tfrac{1}{2}\lambda^\delta < \tfrac{1}{4}\lambda^{\delta/2} + \tfrac{1}{2}\lambda^\delta < \tfrac{3}{4}\lambda^\delta \leq (\tfrac{3}{4})^{1-\delta}|\eta|^{2\delta} \leq |\eta|^{2\delta},$$

where we used $\lambda \leq |\eta|^2 + L < |\eta|^2 + \frac{1}{4}\lambda$ and therefore $\lambda \leq \frac{4}{3}|\eta|^2$.

This shows that $A(\lambda, \lambda^\delta) \cap S_\beta \neq \emptyset$, for some $\beta \neq 0$ such that $|\beta| < \lambda^{\delta/2}$, which in turn implies $A(\lambda, \lambda^\delta) \cap S(\lambda) \neq \emptyset$, contradicting that $\lambda \in \Lambda_g$. So it follows that $|\beta| \geq \lambda^{\delta/2}$. □

The following key Lemma will be used in the computation of the fourth moment.

Lemma 3.2. *Let $\lambda \in \Lambda_g$, $\lambda^{\delta/2} > 2$ and put $L = L(\lambda) := \frac{1}{20}\min(a, 1/a)\lambda^{\delta/2}$. For $\xi, \eta \in A(\lambda, L)$ distinct, the equation*

$$\xi - \eta = \eta' - \xi', \quad \xi', \eta' \in A(\lambda, L) \tag{3.4}$$

has only the trivial solutions

$$(\xi', \eta') = \begin{cases} (\eta, \xi) \\ \\ (-\xi, -\eta). \end{cases} \tag{3.5}$$

Proof. We define the annulus centered at $\omega \in \mathbb{R}^2$ by

$$\mathcal{A}(\omega) = \mathcal{A}(\omega, L) = \left\{ x \in \mathbb{R}^2 \mid \left| |x - \omega|^2 - \lambda \right| < L \right\}$$

and denote $\mathcal{A} = \mathcal{A}(0)$, $\mathcal{B} = \mathcal{A} \cap \mathcal{L}$. Let $\eta, \xi \in \mathcal{B}$ and denote $\beta = \eta - \xi$.

We consider the set

$$\mathcal{S}(\beta) = \left\{ (\eta', \xi') \in \mathcal{B} \times \mathcal{B} \mid \eta' - \xi' = \beta \right\} \tag{3.6}$$

and prove that

$$\mathcal{S}(\beta) = \left\{ (\eta, \xi), (-\xi, -\eta) \right\}.$$

First of all we have from Lemma 3.1 that $|\xi - \eta|, |\xi + \eta| \geq \lambda^{\delta/2}$. Also note that any element (η', ξ') of $\mathcal{S}(\beta)$ satisfies

$$\lambda - L < |\eta'|^2 < \lambda + L$$

and

$$\lambda - L < |\xi'|^2 = |\eta' - \beta|^2 < \lambda + L$$

and thus η' is constrained to lie in $\mathcal{A} \cap \mathcal{A}(\beta) \cap \mathcal{L}$. After changing coordinates by a rotation around the origin we may assume that β is horizontal.

We next show that the intersection of the two annuli cannot have a single connected component. To see this let $R = \sqrt{\lambda + L}$, $r = \sqrt{\lambda - L}$ and note that the case of a single connected component implies the inequality

$$\sqrt{\lambda - L} = r \leq \frac{1}{2}|\beta|.$$

Suppose, for a contradiction, that this inequality holds. Then

$$\frac{1}{4}|\beta|^2 + \frac{1}{4}|\xi+\eta|^2 = \frac{1}{4}|\xi-\eta|^2 + \frac{1}{4}|\xi+\eta|^2 = \frac{1}{2}(|\xi|^2 + |\eta|^2) \leq R^2 = \lambda + L.$$

These two inequalities imply, on recalling our assumption $L = \frac{1}{20}\min(a, 1/a)\lambda^{\delta/2}$

$$\frac{1}{4}|\eta+\xi|^2 \leq \lambda + L - \frac{1}{4}|\beta|^2 \leq 2L < \frac{1}{2}\lambda^{\delta/2}$$

and thus $|\eta + \xi| < \sqrt{2}\lambda^{\delta/4}$. But, as we saw above, our assumption $\lambda \in \Lambda_g$ implies $|\eta + \xi| \geq \lambda^{\delta/2}$, which contradicts the assumption $\lambda^{\delta/2} > 2$.

The case of two connected components. By the above argument, the set

$$\mathcal{A} \cap \mathcal{A}(\beta) =: \mathcal{D}(\eta) \cup \mathcal{D}(-\xi)$$

is thus the union of two approximate parallelograms containing η and $-\xi$ respectively (cf. Figure 1.)

Finding the solutions. We introduce coordinates x, y such that the annulus $\mathcal{A}$ is centered at $(x, y) = (0, 0)$ and $\mathcal{A}(\beta)$ is centered at $(x, y) = (|\beta|, 0)$. We compute the coordinates of the vertices ω_1, ω_2, ν_1, ν_2 of $\mathcal{D}(\eta)$ in order to calculate the distances $h = |\omega_1 - \omega_2|$ and $w = |\nu_1 - \nu_2|$ (cf. Figure 1).

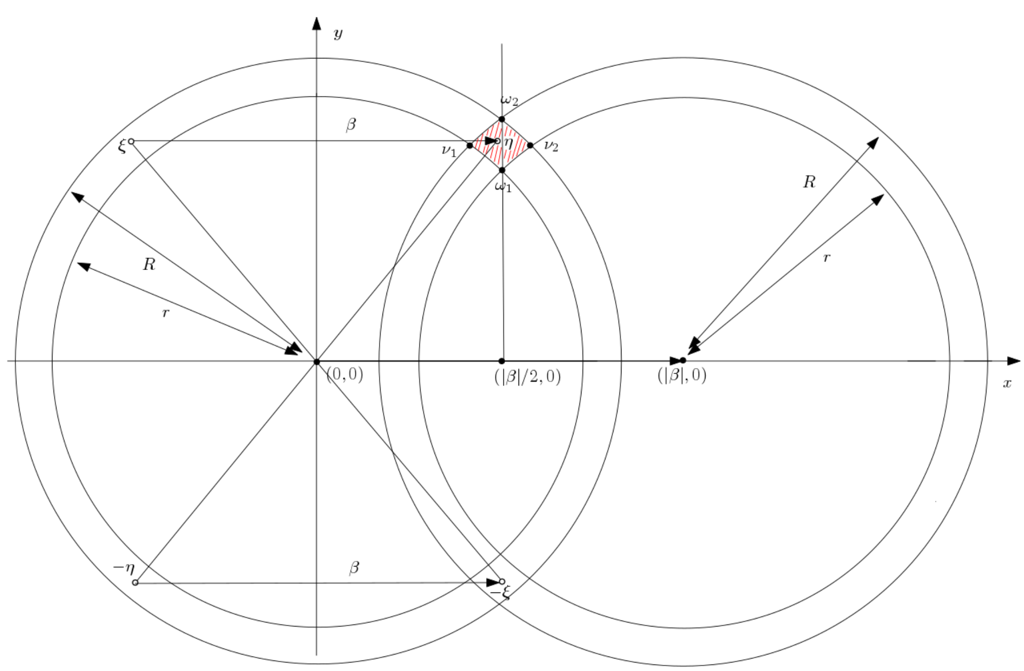

Figure 1. The intersection of the two annuli $\mathcal{A}(0)$ and $\mathcal{A}(\beta)$. In order to calculate the diameter of the approximate parallelogram $\mathcal{D}(\eta)$ with the vertices ω_1, ω_2, ν_1, ν_2 we have applied a rotation and introduced cartesian coordinates x, y such that $\beta = (0, |\beta|)$ in these new coordinates.

We aim for a bound on the diameter of $\mathcal{D}$ which is smaller than the minimal distance between two lattice points, so that $\mathcal{D}$ may contain at most one lattice point. To this end, we observe that $\mathcal{D} \subset \mathcal{R} = [x_-, x_+] \times [y_r, y_R]$, where x_-, x_+ are the x-coordinates of the points ν_1, ν_2 and y_r, y_R are the y-coordinates of the points ω_1, ω_2. We then bound the diameter of $\mathcal{R}$.

By solving the equations

$$x^2 + y^2 = r_1^2 \qquad (x - |\beta|)^2 + y^2 = r_2^2$$

for the cases $r_1 = r, R$ and $r_2 = r, R$, we obtain

$$\omega_1 = (\tfrac{1}{2}|\beta|, y_r), \quad \omega_2 = (\tfrac{1}{2}|\beta|, y_R)$$

where $y_r = \sqrt{r^2 - \frac{1}{4}|\beta|^2}$ and $y_R = \sqrt{R^2 - \frac{1}{4}|\beta|^2}$. It follows that

$$h = |\omega_1 - \omega_2| = y_R - y_r = \sqrt{R^2 - \tfrac{1}{4}|\beta|^2} - \sqrt{r^2 - \tfrac{1}{4}|\beta|^2}$$

and therefore (recall $R = \sqrt{\lambda + L}$ and $r = \sqrt{\lambda - L}$)

$$h = \frac{R^2 - r^2}{\sqrt{R^2 - \frac{1}{4}|\beta|^2} + \sqrt{r^2 - \frac{1}{4}|\beta|^2}} = \frac{2L}{\sqrt{\lambda + L - \frac{1}{4}|\beta|^2} + \sqrt{\lambda - L - \frac{1}{4}|\beta|^2}}.$$

Furthermore, by symmetry we have

$$\nu_1 = (x_-, y_\nu), \quad \nu_2 = (x_+, y_\nu)$$

for some $y_\nu > 0$ and $x_\pm = \frac{1}{2}|\beta| \pm \Delta_\nu$ for some $\Delta_\nu > 0$. We then have

$$x_+ - x_- = |\nu_1 - \nu_2| = 2\Delta_\nu.$$

In order to determine Δ_ν we solve the system of equations

$$x_-^2 + y_\nu^2 = r^2, \quad x_+^2 + y_\nu^2 = R^2$$

which implies

$$x_+^2 - x_-^2 = R^2 - r^2.$$

It follows that $2|\beta|\Delta_\nu = R^2 - r^2 = 2L$. In summary, using that $|\beta| = |\eta - \xi| \geq \lambda^{\delta/2}$, we find that

$$h = \frac{2L}{\sqrt{\lambda + L - \frac{1}{4}|\beta|^2} + \sqrt{\lambda - L - \frac{1}{4}|\beta|^2}} \text{ and } w = \frac{2L}{|\beta|} < 2\frac{L}{\lambda^{\delta/2}},$$

respectively. Now, since $0 < L < \frac{1}{4\sqrt{2}} \min(a, 1/a)\lambda^{\delta/2}$, it follows that $w < \min(a, 1/a)/\sqrt{2}$ and

$$h \leq \frac{2L}{\sqrt{R^2 - \frac{1}{4}|\beta|^2}} \leq \frac{4L}{\lambda^{\delta/2}} < \frac{\min(a, 1/a)}{\sqrt{2}}$$

since $\frac{1}{4}|\beta|^2 + \frac{1}{4}|\xi+\eta|^2 = \frac{1}{2}(|\xi|^2+|\eta|^2) \le R^2$ and $|\xi+\eta| \ge \lambda^{\delta/2}$.

Hence $\operatorname{diam}\mathcal{D}(\eta) \le \operatorname{diam}\mathcal{R}(\eta) = \sup_{x,y\in\mathcal{R}(\eta)}|x-y| \le \sqrt{2}\max\{w,h\} < \min(a,1/a)$ and, therefore, η is the only lattice point in $\mathcal{D}(\eta)$.

By symmetry it follows that $\mathcal{D}(-\xi)$ also contains only the lattice point $-\xi$. This proves the claim. □

3.3. **Evaluating the fourth moment.** Recall the truncated Green's function

$$G_{\lambda,L}(x) = \sum_{\xi\in A(\lambda,L)} c_\lambda(\xi)e^{i\xi\cdot x}, \quad c_\lambda(\xi) = \frac{1}{|\xi|^2-\lambda}. \tag{3.7}$$

We evaluate the L^4-norm of the truncated Green's function in terms of its L^2-norm.

Lemma 3.3. *Let $\lambda \in \Lambda_g$ and and put $L = L(\lambda) := \frac{1}{20}\min(a,1/a)\lambda^{\delta/2}$. Then (recall that $\mathbb{E}(f) = \int_{\mathbb{T}^2} f(x)d\mu(x)$)*

$$\mathbb{E}\left(\frac{G_{\lambda,L}^4}{\|G_{\lambda,L}\|_2^4}\right) = 3 - 2\frac{\sum_{\xi\in A(\lambda,L)} c_\lambda(\xi)^4}{\|G_{\lambda,L}\|_2^4}.$$

Proof. Let

$$a_\xi = \begin{cases} \frac{1}{|\xi|^2-\lambda}, \text{ if } ||\xi|^2-\lambda| < L \\ \\ 0, \text{ otherwise;} \end{cases}$$

clearly $a_\xi = a_{-\xi}$. Now

$$\begin{aligned} \|G_{\lambda,L}\|_4^4 &= \sum_{\substack{\xi_1,\xi_2,\eta_1,\eta_2\in\mathcal{L} \\ \xi_1+\xi_2=\eta_1+\eta_2}} a_{\xi_1}a_{\xi_2}a_{\eta_1}a_{\eta_2} \\ &= \sum_{0=\xi_1-\eta_1=\eta_2-\xi_2} a_{\xi_1}a_{\xi_2}a_{\eta_1}a_{\eta_2} \\ &\quad + \sum_{\substack{\beta\neq 0 \\ \beta=\xi_1-\eta_1=\eta_2-\xi_2}} a_{\xi_1}a_{\xi_2}a_{\eta_1}a_{\eta_2}. \end{aligned} \tag{3.8}$$

The first sum can be rewritten as

$$\sum_{\xi_1,\xi_2} a_{\xi_1}^2 a_{\xi_2}^2 = \|G_{\lambda,L}\|_2^4. \tag{3.9}$$

With regard to the second sum let us consider the solutions of the equation

$$\eta_2 - \xi_2 = \beta$$

where

$$0 \neq \beta = \xi_1 - \eta_1$$

and

$$\xi_1,\xi_2,\eta_1,\eta_2 \in A(\lambda,L).$$

Our assumption that $\lambda \in \Lambda_g$, together with Lemma 3.4, implies that the only solutions are of the form

$$(\xi_2, \eta_2) = \begin{cases} (\eta_1, \xi_1) \\ \\ (-\xi_1, -\eta_1). \end{cases} \tag{3.10}$$

Hence, we can rewrite the second sum as

$$2 \sum_{\xi_1, \eta_1, \xi_1 \neq \eta_1} a_{\eta_1}^2 a_{\xi_1}^2 = 2\, \|G_{\lambda,L}\|_2^4 - 2 \sum_{\xi} a_\xi^4. \tag{3.11}$$

The result follows. □

We have the following Lemma which shows that the 4th moment cannot be Gaussian, unless the Laplace spectrum has unbounded multiplicities.

Lemma 3.4. *Given $\epsilon \in (0,1)$ there exists a subset of Λ, of density $1-\epsilon$, and a constant $C_\epsilon > 0$ such that for all sufficiently large λ in the said subsequence, we have*

$$\frac{\sum_{\xi \in A(\lambda,L)} c_\lambda(\xi)^4}{\|G_{\lambda,L}\|_2^4} \geq C_\epsilon.$$

Proof. We will use the following convenient notation: for $m \in \mathcal{N}$, let $m_- := \max\{n \in \mathcal{N} : n < m\}$ denote the unperturbed eigenvalue immediately preceeding m, and let λ_m denote the unique perturbed eigenvalue in (m_-, m). We claim that there exists a subsequence of Λ of the form $\{\lambda_m\}_{m \in \mathcal{N}'}$, where $\mathcal{N}'$ is of density $1-\epsilon$ in $\mathcal{N}$, such that a positive proportion of the L^2-norm is captured by a finite set of frequencies in the sense that for $I_m := \mathcal{N} \cap [m_- - 3, m+3]$, we have

$$\sum_n \frac{r_{\mathcal{L}}(n)}{(n-\lambda_m)^2} \ll_\epsilon \sum_{n \in I_m} \frac{r_{\mathcal{L}}(n)}{(n-\lambda_m)^2}$$

and, as $m \to \infty$ along this subsequence, that $|I_m|$ remains bounded.

Let us explain the construction in more detail. In view of the remarks after Lemma 4.2 in [12] (here the Diophantine condition is again crucial) we may construct a subsequence $\mathcal{N}''$ of density $1-\epsilon$ such that for $m \in \mathcal{N}''$ we have

$$\sum_{|n-m|>3} \frac{r_{\mathcal{L}}(n)}{(n-m)^2} \leq F_\epsilon,$$

$$\#\{0 < |n-m| \leq 3\} \leq E_\epsilon$$

and

$$|m - \lambda_m| \leq G_\epsilon,$$

for some numbers $E_\epsilon, F_\epsilon, G_\epsilon > 0$.

We then have

(3.12)

$$\sum_n \frac{r_{\mathcal{L}}(n)}{(n-\lambda_m)^2} = \sum_{n\in I_m} \frac{r_{\mathcal{L}}(n)}{(n-\lambda_m)^2} + \sum_{n\notin I_m} \frac{r_{\mathcal{L}}(n)}{(n-\lambda_m)^2}$$
$$< \sum_{n\in I_m} \frac{r_{\mathcal{L}}(n)}{(n-\lambda_m)^2} + \sum_{|n-m_-|>3} \frac{r_{\mathcal{L}}(n)}{(n-m_-)^2} + \sum_{|n-m|>3} \frac{r_{\mathcal{L}}(n)}{(n-m)^2}$$

where we used the inequalities

$$\sum_{\substack{|n-m_-|>3\\ n<m_-}} \frac{r_{\mathcal{L}}(n)}{(n-\lambda_m)^2} < \sum_{\substack{|n-m_-|>3\\ n<m_-}} \frac{r_{\mathcal{L}}(n)}{(n-m_-)^2}$$

and

$$\sum_{\substack{|n-m|>3\\ n>m}} \frac{r_{\mathcal{L}}(n)}{(n-\lambda_m)^2} < \sum_{\substack{|n-m|>3\\ n>m}} \frac{r_{\mathcal{L}}(n)}{(n-m)^2}.$$

So we may define a subsequence $\mathcal{N}' \subset \mathcal{N}''$ (of density at least $1-2\epsilon$) consisting of those $m \in \mathcal{N}''$ such that also $m_- \in \mathcal{N}''$ holds. For $m \in \mathcal{N}'$ we have

$$\sum_n \frac{r_{\mathcal{L}}(n)}{(n-\lambda_m)^2} < \sum_{n\in I_m} \frac{r_{\mathcal{L}}(n)}{(n-\lambda_m)^2} + 2F_\epsilon.$$

Now, as the term corresponding to $n=m$ in the sum $\sum_{n\in I_m} \frac{r_{\mathcal{L}}(n)}{(n-\lambda_m)^2}$ is bounded below by $1/G_\epsilon^2$, we find on multiplying by $F_\epsilon G_\epsilon^2$, that

$$F_\epsilon \leq F_\epsilon G_\epsilon^2 \sum_{n\in I_m} \frac{r_{\mathcal{L}}(n)}{(n-\lambda_m)^2}.$$

Thus

$$\sum_n \frac{r_{\mathcal{L}}(n)}{(n-\lambda_m)^2} \leq (1+2F_\epsilon G_\epsilon^2) \sum_{n\in I_m} \frac{r_{\mathcal{L}}(n)}{(n-\lambda_m)^2}$$

and we note that $\#\{n \in I_m\} \leq 2E_\epsilon$. This implies

(3.13)

$$\left(\sum_{|n-\lambda_m|\leq L} \frac{r_{\mathcal{L}}(n)}{(n-\lambda_m)^2}\right)^2 \leq \left(\sum_n \frac{r_{\mathcal{L}}(n)}{(n-\lambda_m)^2}\right)^2 \ll_\epsilon \left(\sum_{n\in I_m} \frac{r_{\mathcal{L}}(n)}{(n-\lambda_m)^2}\right)^2$$
$$\leq |\{n\in I_m\}| \sum_{n\in I_m} \frac{r_{\mathcal{L}}(n)^2}{(n-\lambda_m)^4} \ll_\epsilon \sum_{|n-\lambda_m|\leq L} \frac{r_{\mathcal{L}}(n)}{(n-\lambda_m)^4} = \sum_{\xi\in A(\lambda_m,L)} c_{\lambda_m}(\xi)^4$$

where we used Cauchy-Schwarz and the fact that the multiplicities $r_{\mathcal{L}}(n)$ are bounded (as the aspect ratio of $\mathcal{L}$ is irrational; note that $\mathcal{L}$ and $\mathcal{L}_0$ have the same aspect ratio. □

It is a simple consequence of the Lemma above that if the multiplicities in the unperturbed Laplace spectrum are bounded, as is the case for Šeba's billiard in the irrational aspect ratio case, then (provided that the Diophantine condition holds) one can construct an essentially full density subsequence of new eigenvalues such that the 4th moment does not converge to the Gaussian 4th moment, as the eigenvalue tends to infinity.

Corollary 3.5. *Denote by $g_{\lambda,L}$ the L^2-normalized, truncated Green's function on an irrational torus with Diophantine aspect ratio, and put $L = L(\lambda) := \frac{1}{20}\min(a,1/a)\lambda^{\delta/2}$. For any $\epsilon > 0$, there exists $C_\epsilon > 0$ and a subsequence of Λ, of density $1-\epsilon$, such that*

$$1 \leq \mathbb{E}(g_{\lambda,L}^4) \leq 3 - 2C_\epsilon$$

as $\lambda \to \infty$ along the said subsequence.

Proof. We recall that there exists a full density subsequence Λ_g such that for $\lambda \in \Lambda_g$ we have

$$\frac{\|G_{\lambda,L}\|_4^4}{\|G_{\lambda,L}\|_2^4} = 3 - 2\frac{\sum_{\xi\in A(\lambda,L)} c_\lambda(\xi)^4}{\|G_{\lambda,L}\|_2^4}.$$

We also note

$$\sum_{\xi\in A(\lambda,L)} c_\lambda(\xi)^4 \leq \left(\sum_{\xi\in A(\lambda,L)} c_\lambda(\xi)^2\right)^2 = \|G_{\lambda,L}\|_2^4.$$

At the same time Lemma 3.4 shows that for any $\epsilon > 0$ there exists $C_\epsilon > 0$ and a subsequence of density $1-\epsilon$ such that

$$1 \geq \frac{\sum_{\xi\in A(\lambda,L)} c_\lambda(\xi)^4}{\|G_{\lambda,L}\|_2^4} \geq C_\epsilon$$

More precisely, if we take λ belonging to the intersection of the two subsequences (a subsequence of density at least $1-2\epsilon$) we have

$$1 \leq \mathbb{E}(g_{\lambda,L}^4) \leq 3 - 2C_\epsilon.$$

□

In order to conclude the proof of the theorem we need the following approximation.

Lemma 3.6. *Let $L = L(\lambda) := \frac{1}{20}\min(a,1/a)\lambda^{\delta/2}$. There exists a subsequence $\{\lambda_{j_k}\}_k$ of $\Lambda = \{\lambda_j\}_j$ of density at least $1-\epsilon$ s. t.*

$$\lim_{k\to\infty}\left|\frac{\|G_{\lambda_{j_k},L}\|_4^4}{\|G_{\lambda_{j_k},L}\|_2^4} - \frac{\|G_{\lambda_{j_k}}\|_4^4}{\|G_{\lambda_{j_k}}\|_2^4}\right| = 0.$$

Proof. Recall $\mathbb{E}(g_{\lambda,L}^4) = \|G_{\lambda,L}\|_4^4/\|G_{\lambda,L}\|_2^4$.

There is a subsequence $\Lambda_1 \subset \Lambda$ (cf. Corollary 3.5) of density at least $1-\epsilon$ s.t. for $\lambda \in \Lambda_1$ we have that $\|G_{\lambda,L}\|_4/\|G_{\lambda,L}\|_2$ is bounded from both above and below. Moreover, there is another subsequence $\Lambda_2 \subset \Lambda$ of density at least $1-\epsilon$ s.t. for $\lambda \in \Lambda_2$ we have $\|G_\lambda\|_2 \gg_\epsilon 1$ (by the same argument as in [12], taking $G = \epsilon^{-1}$ on p. 16).

Let us denote $\Lambda_1 \cap \Lambda_2 = \{\lambda_{j_k}\}_{k=0}^{+\infty}$ which is a subsequence of density at least $1-2\epsilon$.

It is sufficient to show that

$$\Big|\frac{\|G_{\lambda_{j_k},L}\|_4}{\|G_{\lambda_{j_k},L}\|_2} - \frac{\|G_{\lambda_{j_k}}\|_4}{\|G_{\lambda_{j_k}}\|_2}\Big| \to 0.$$

We first note that

$$\begin{aligned} &\Big|\frac{\|G_{\lambda_{j_k},L}\|_4}{\|G_{\lambda_{j_k},L}\|_2} - \frac{\|G_{\lambda_{j_k}}\|_4}{\|G_{\lambda_{j_k}}\|_2}\Big| \\ &\leq \|G_{\lambda_{j_k}}\|_2^{-1}\Big|\|G_{\lambda_{j_k}}\|_4 - \|G_{\lambda_{j_k},L}\|_4\Big| \\ &\quad + \|G_{\lambda_{j_k}}\|_2^{-1}\frac{\|G_{\lambda_{j_k},L}\|_4}{\|G_{\lambda_{j_k},L}\|_2}\Big|\|G_{\lambda_{j_k}}\|_2 - \|G_{\lambda_{j_k},L}\|_2\Big|. \end{aligned} \tag{3.14}$$

Thus, using that $\|G_{\lambda_{j_k}}\|_2 \gg_\epsilon 1$, as well as Corollary 3.5, and finally the reverse triangle inequality $|\|f\|_p - \|g\|_p| \leq \|f-g\|_p$ for $p = 2, 4$, we find that the right hand side of (3.14) is

$$\begin{aligned} \ll_\epsilon \|G_{\lambda_{j_k}} - G_{\lambda_{j_k},L}\|_4 + \|G_{\lambda_{j_k}} - G_{\lambda_{j_k},L}\|_2 &\ll \|G_{\lambda_{j_k}} - G_{\lambda_{j_k},L}\|_4 \\ &\ll L^{-1/4+o(1)} \longrightarrow 0, \quad \text{as } \lambda \to \infty, \end{aligned}$$

using that $\|f\|_2 \ll \|f\|_4$ for any $f \in L^4(\mathbb{T}^2)$, together with Lemma 2.1. $\square$

If we take λ belonging to the subsequence of Lemma 3.6, then we have for λ sufficiently large

$$\big|\|g_\lambda\|_4^4 - \|g_{\lambda,L}\|_4^4\big| = o(1)$$

and

$$\|g_{\lambda,L}\|_4^4 \in [1, 3-2C_\epsilon].$$

Hence, it follows (recall $\mathbb{E}(g_\lambda^4) = \|g_\lambda\|_4^4$)

$$1 + o(1) \leq \mathbb{E}(g_\lambda^4) \leq 3 - 2C_\epsilon + o(1).$$

Appendix A. Dirichlet boundary conditions

In [14] Šeba discussed irrational aspect ratio rectangles with Dirichlet boundary conditions rather than rectangular tori. In particular, this means that the wave functions and the spectrum depend on the position of the scatterer. We briefly discuss here how our results can easily be extended to this setting.

Let us denote the position of the scatterer by $y = (y_1, y_2)$, with the irrationality conditions $y_1 \notin 2\pi a\mathbb{Q}$ and $y_2 \notin (2\pi/a)\mathbb{Q}$. Denote $\mathcal{L}^+ = \{\xi \in \mathcal{L} \mid \xi_1, \xi_2 > 0\}$. The new eigenfunctions are then of the form

$$G_\lambda(x) = \sum_{\xi \in \mathcal{L}^+} c_\lambda(\xi)\psi_\xi(y)\psi_\xi(x) \tag{A.1}$$

where $\psi_\xi(x) = \sin(\xi_1 x_1)\sin(\xi_2 x_2)$. We note that the summation can easily be written over $\mathcal{L}$:

$$G_\lambda(x) = -\frac{1}{4}\sum_{\xi \in \mathcal{L}} c_\lambda(\xi)\psi_\xi(y)\chi(\xi)e^{i\xi\cdot x},$$

where $\chi(\xi) = \operatorname{sgn}(\xi_1)\operatorname{sgn}(\xi_2)$.

In order to prove the analogue of Theorem 1.1 we require analogues of the argument for the L^4-convergence in section 2.1, as well as the Lemmas 2.1, 3.3, 3.4 and 3.6.

The arguments of section 2.1 and Lemma 2.1 work analogously because of the bound $|\psi_\xi(y)| \leq 1$.

The proof of Lemma 3.3 works exactly the same way, as it only depends on the structure of the set of lattice points in the annulus $A(\lambda, L)$. In the case of Dirichlet boundary conditions it yields

$$\mathbb{E}\left(\frac{G_{\lambda,L}^4}{\|G_{\lambda,L}\|_2^4}\right) = 3 - 2\frac{\sum_{\xi \in A(\lambda,L)} c_\lambda(\xi)^4\psi_\xi(y)^4}{\|G_{\lambda,L}\|_2^4}$$

and

$$\|G_{\lambda,L}\|_2^2 = \sum_{\xi \in A(\lambda,L)} c_\lambda(\xi)^2\psi_\xi(y)^2$$

The analogue of Lemma 3.4 can then be readily obtained by replacing $r_{\mathcal{L}}(n)$ with the function

$$r_{\mathcal{L}}(n, y) = \sum_{|\xi|^2 = n} \psi_\xi(y)^2 \leq r_{\mathcal{L}}(n),$$

provided we can construct a (large density) subsequence of Λ such that

$$\sum_{n \in I_m} \frac{r_{\mathcal{L}}(n)}{(n - \lambda_m)^4} \ll \sum_{n \in I_m} \frac{r_{\mathcal{L}}(n, y)}{(n - \lambda_m)^4}.$$

To do this, we define the "bad" set of eigenvalues

$$B = \{\lambda_k \in \Lambda' \mid \exists n \in \mathcal{N} \cap I_k : \ |\psi_\xi(y)| < \delta, |\xi|^2 = n\}$$

where Λ' denotes the subsequence of eigenvalues such that $\#\{n \in I_m\}$ remains bounded. For $\epsilon > 0$ we may construct Λ' of density at least $1 - \epsilon$ such that $\#\{n \in I_m\} \leq N(\epsilon)$.

We can now estimate the cardinality of the bad set, because for each $n \in \mathcal{N}$ such that $|\psi_\xi(y)| < \delta$ for $|\xi|^2 = n$ there exists only a finite number K_ϵ of $\lambda_k \in \Lambda'$ with $n \in I_k$. At the same time, as the irrationality condition on y implies that $\xi \cdot y$ equidistributes modulo 2π as ξ ranges over lattice points in

$\mathcal{L}$ such that $|\xi|^2 \leq T$, and thus (keeping in mind that if $\xi, \xi' \in \mathcal{L}$ and $|\xi| = |\xi'|$ then the components agree up to sign and therefore $|\psi_\xi(y)| = |\psi_{\xi'}(y)|$) we find that

$$\#\{n \in \mathcal{N} : n \leq T, |\psi_\xi(y)| < \delta, |\xi|^2 = n\} = O(\delta T)$$

so that $|B| = O(\delta T K_\epsilon)$ and we can make δ small enough in terms of ϵ such that the subsequence of bad eigenvalues is of density less than ϵ. Thus, after excluding the bad eigenvalues we obtain a subsequence of density at least $1 - 2\epsilon$.

The proof of Lemma 3.6, however, requires a lower bound for $\|G_{\lambda,L}\|_2$. The above argument (or see the appendix of [12]), also shows that if y is generic (in particular that the coordinates y_1, y_2 are irrational in the above sense) there exists a subsequence of Laplace eigenvalues of arbitrarily high density such that for $|\xi|^2 = n$ we have $\liminf_{n\to\infty} |\psi_\xi(y)| > 0$ along the said subsequence. This yields the lower bound $\|G_{\lambda,L}\|_2 \gg_\epsilon 1$.

In conclusion, the above argument rules out a Gaussian fourth moment (as well as Gaussian value distribution) also for new eigenfunctions of point scatterers with Dirichlet boundary conditions on rectangles, provided the position y of the scatterer is generic (and assuming the previous Diophantine condition on the aspect ratio.)

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