AREA CORRELATIONS RELATED TO LATTICE POINTS IN DISCS

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Abstract. Motivated by the Lester-Wigman vanishing area correlation conjecture for lattice points near the boundary of circles of growing radius, we investigate the dynamics of circle flows on tori (which are related to the motion of a charged particle in a magnetic field on a torus.) We show that an analogue of the vanishing correlation conjecture holds in this setting, i.e., we have "mixing for the area observable" despite the flow being essentially integrable. We also determine the probability density function of the areas, in global as well as local regimes.

1. Introduction

Given $R > 0$, let

$$B(R) := \{x \in \mathbb{R}^2 : |x| \leq R\}$$

denote the radius R ball with center at $(0, 0)$, let the circle $C(R)$ denote the boundary of $B(R)$, and let $\Lambda := \mathbb{Z}^2$ denote the standard lattice in $\mathbb{R}^2$. A standing assumption is that R is chosen so that $C(R) \cap \Lambda = \emptyset$; it suffices to avoid a countable subset of radii for this to hold. The "half open" unit square $S := [0, 1) \times [0, 1)$ is a fundamental domain for the action of Λ on $\mathbb{R}^2$. For $\lambda \in \Lambda$ define $S_\lambda := S + \lambda$ and note the *disjoint* union $\cup_{\lambda \in \Lambda} S_\lambda = \mathbb{R}^2$.

For $R > 0$ define

$$\Lambda(R) := \{\lambda \in \Lambda : S_\lambda \cap C(R) \neq \emptyset\}$$

and for $\lambda \in \Lambda(R)$, let $A(\lambda)$ denote the area of $S_\lambda \cap B(R)$. The main result of this paper is that the area correlation, for $k \in \mathbb{Z}$,

$$\mathrm{Corr}(R, k) := \frac{1}{|\Lambda(R)|} \sum_{\lambda_j \in \Lambda(R)} (A(\lambda_j) - 1/2) \cdot (A(\lambda_{j+k}) - 1/2)$$

vanishes as $R \to \infty$ provided $|k| = O(R^{1-\epsilon})$ grows with R, where $\{\lambda_1, \lambda_2, \ldots\}$ denotes the elements of $\Lambda(R)$ ordered by increasing argument (after identifying $\mathbb{R}^2$ with $\mathbb{C}$.) We subtract by $1/2$ since the mean area is $1/2 + o(1)$ as $R \to \infty$, cf. Proposition 2. Further, here and in what follows we use the convention of cyclic wraparound in case $j+k > |\Lambda(R)|$ or $j+k < 1$ (in case $k<0$).

To prove that the (global) area correlation vanishes we work locally. Given an interval $I = I(R) \subset [0, 2\pi]$, allowed to shrink as R grows, we define

$$\Lambda(R, I) := \{\lambda \in \Lambda(R) : \arg(\lambda) \in I\},$$

and study the local area correlations

$$\mathrm{Corr}(R,k,I) := \frac{1}{|\Lambda(R,I)|} \sum_{\lambda_j \in \Lambda(R,I)} (A(\lambda_j) - 1/2) \cdot (A(\lambda_{j+k}) - 1/2)$$

In fact, when[1] $k \asymp R^{1-\epsilon}$ we must allow $|I| = o(1/k) = o(1/R^{1-\epsilon})$ for "most" such short intervals I — a key issue is to avoid intervals for which the slope of the tangent line of the circle is well approximated by a rational number of low height.

Theorem 1. *Given $\epsilon > 0$, provided*

$$k = k(R) = O(R^{1-\epsilon})$$

and

$$|I| > (\log\log Q)/\log Q \text{ with } Q = \min(\log R, \sqrt{k}/\log k),$$

we have

$$\mathrm{Corr}(R,k,I) = o(1),$$

as $k, R \to \infty$.

There is a tradeoff between taking k large and I having small size. Our primary interest is taking k essentially as large as possible, but the methods allow us to take, say, $|I| \gg (\log R)^8/\sqrt{R}$ at the expense of having to further restrict the size of k. We note that there is a natural barrier near $|I| = R^{-1/2}$: for $|I| = o(R^{-1/2})$ the local mean area can significantly deviate from $1/2$ (e.g. at the lower part of the circle.)

For comparison, we also prove some simpler statistics related to the areas $A(\lambda_j)$. We define the (global) mean area as

$$\mathrm{Mean}(R) := \frac{1}{|\Lambda(R)|} \sum_{\lambda_j \in \Lambda(R)} A(\lambda_j),$$

and the (global) variance as

$$\mathrm{Var}(R) := \frac{1}{|\Lambda(R)|} \sum_{\lambda_j \in \Lambda(R)} (A(\lambda_j) - \mathrm{Mean}(R))^2 .$$

We prove the following asymptotics for the global mean and variance.

[1]By $k \asymp R^{1-\epsilon}$ we mean that $k \ll R^{1-\epsilon} \ll k$.

Proposition 2. *We have*

$$\mathrm{Mean}(R) = \frac{1}{2} + o(1),$$

and

$$\mathrm{Var}(R) = c_{\mathrm{glob}} + o(1), \quad c_{\mathrm{glob}} := \frac{13}{60} - \frac{1}{12}\log(1+\sqrt{2}) - \frac{1}{30\sqrt{2}} = 0.119\ldots > 0,$$

as $R \to \infty$.

In fact, we compute the asymptotic for the local mean and variance where we allow $|I|$ to shrink in two regimes. Namely, for $|I| > 1/R^{\frac{1}{2}-\epsilon}$ (and any small fixed $\epsilon > 0$) as well as for much shorter intervals at "generic" position; here generic is related to Diophantine properties of $\tan(\arg(\lambda))$), see Section 6. While the expectation is locally constant[2] for $|I| > 1/R^{\frac{1}{2}-\epsilon}$, the variance is not and depends on $\arg(\lambda)$.

We also determine the cumulant and the probability density function of the areas $A(\lambda)$ for $\lambda \in \Lambda(R)$.

Proposition 3. *For* $t \in [0,1]$, *define*

$$F_R(t) := \frac{1}{|\Lambda(R)|} \#\{\lambda \in \Lambda(R) : \ A(\lambda) \leq t\}.$$

Then, as $R \to \infty$,

$$F_R(t) \longrightarrow \int_0^t f(a)\, da$$

where the density function f, *for* $0 < a < 1$ *and* $z_a = \min(a, 1-a)$, *is*

$$f(a) = \frac{2z_a}{\sqrt{1+4z_a^2}} + \frac{1}{\sqrt{2z_a}} \int_{2z_a}^{1} \frac{\sqrt{m}}{(1+m^2)^{3/2}}\, dm.$$

We note that $f(1-a) = f(a)$, and that the above integral can be written as an elliptic integral of the second kind. Interestingly, the density function has two (integrable) singularities at $a = 0, 1$, cf. Figure 1.

[2]We remark that for $|I| = o(\sqrt{R})$ the local mean is generically $1/2 + o(1)$, but for non-generic placements of I the local mean might deviate from $1/2$.

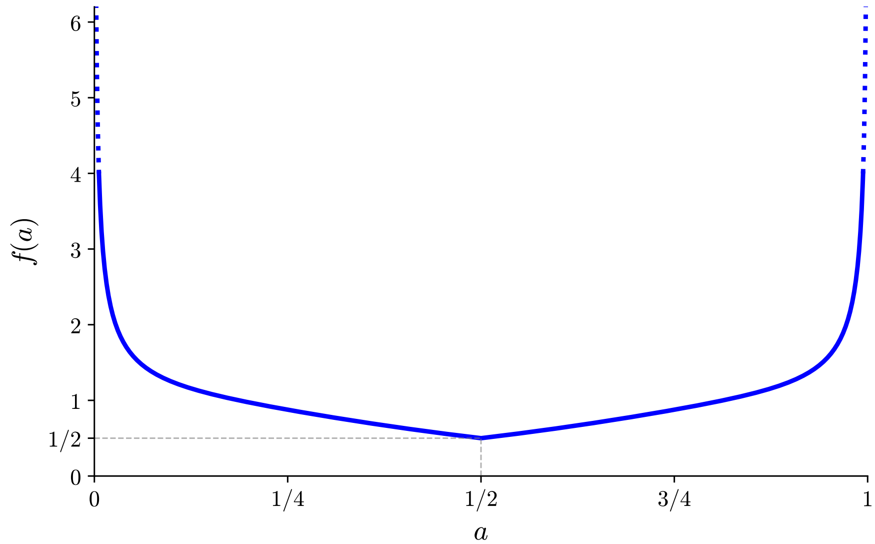

FIGURE 1. Density function $f(a)$

The singularity near $a = 0$ is asymptotically given by

$$f(a) \sim \frac{\Gamma^2\left(\frac{3}{4}\right)}{2\sqrt{2\pi}} \frac{1}{\sqrt{a}},$$

and similarly for $1 - a$ small.

1.1. **Discussion.** Our investigations were inspired by work of Lester and Wigman who in [5], motivated by a probabilistic folklore heuristic for Hardy's conjectured $O(R^{1/2+\epsilon})$ error term for the Gauss circle problem (cf. [2]), initiated the study of area correlations of the intersections of $B(R)$ with unit square translates of lattice points inside thin annuli. More precisely, for $R > 0$ large, they considered a different set of lattice points $\tilde{\Lambda}(R) := \{\lambda \in \mathbb{Z}^2 : ||\lambda| - R| < 1/\sqrt{2}\}$, ordered by argument, with associated areas[3] $\tilde{A}_{\lambda_i} := |B(R) \cap ([-1/2, 1/2]^2 + \lambda_i)|$ and showed that the area correlation

$$\widetilde{C_k} = \lim_{R\to\infty} \frac{1}{|\tilde{\Lambda}(R)|} \sum_{i=1}^{|\tilde{\Lambda}(R)|} (\tilde{A}_{\lambda_i} \cdot \tilde{A}_{\lambda_{i+k}} - 1/4)$$

[3]One notable difference from our setup is that $\tilde{A}_{\lambda_i} = 1$ or $\tilde{A}_{\lambda_i} = 0$ occurs for a positive proportion of elements in $\tilde{\Lambda}(R)$; this is due to including translates of squares either lying entirely inside or outside of $B(R)$. Interestingly this introduces some additional randomization compared to our setting (the "extra"/"empty" squares has a reshuffling effect on the ordering of the squares), though it appears difficult to exploit.

exists for each k. They also showed vanishing on average in that $\lim_{K\to\infty}\frac{1}{K}\sum_{k=1}^{K}\widetilde{C}_k = 0$, and also conjectured that $\lim_{k\to\infty}\widetilde{C}_k = 0$.

Our approach is in the spirit of dynamical systems, and can be viewed as a study of a "circle flow" on the torus, which in turn can be given a physical interpretation as the motion of an electron on a torus with a magnetic field; $R \to \infty$ corresponds to a vanishing magnetic field. Now, if the flow (or rather, the Poincare map with respect to the Poincare section given by the sides of the square) was mixing, vanishing area correlations would immediately follow. However, as the circle flow can be well approximated by a linear flow for long times, mixing certainly does not hold, the angle of intersection with vertical and horizontal lines remain essentially constant.

Rather, we exploit a certain "order stability": a line segment, or circular arc, connecting two squares and passing through $k-1$ intermediate squares can be perturbed while preserving the ordering. This, together with an equidistribution result pertaining to very small angular variation of tangent lines allows us to decompose the area correlation sum into parts on which the areas approximately decorrelate. More precisely, the area of the intersection of the first square is essentially constant, while the second area intersections (roughly) uniformly "sweeps through" all areas between zero and one as the angle varies (in very small intervals.) Our setup turns out to somewhat be in the spirit of Gauss' original argument for the circle problem — he divides the squares intersecting the ball into internal squares and boundary squares, and estimates the number of boundary squares in terms of the length of the boundary, c.f. Appendix A for details.

We finally remark that our global mean area estimate $1/2 + o(1)$ implies a slight improvement on Gauss $O(R)$ error term, due to a curious "perfect cancellation" property between the boundary terms of quarter circles, cf. Appendix B for more details. We also show that error $O(R^{1/2+\epsilon})$, for any $\epsilon > 0$, would follow from a certain IID hypothesis on the distribution of $\{A(\lambda)\}_{\lambda\in\Lambda(R)}$.

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2. Preliminaries

2.1. **Lattice points near curves.**

Given a curve $\gamma \subset \mathbb{R}^2$, we say that a lattice point $\lambda \in \Lambda$ is near γ if $\gamma \cap S_\lambda \neq \emptyset$. We begin with some simple observations regarding the number of lattice points near curves, in particular near circular arcs.

Lemma 4. *Let γ be a continuous curve having $v_1, v_2 \in \mathbb{R}^2$ as endpoints. The number of $\lambda \in \Lambda$ such that $\gamma \cap S_\lambda \neq \emptyset$ is $\gg |v_1 - v_2|$.*

Further, if $\gamma(t) = (x(t), y(t))$ and both coordinate functions $x(t)$ and $y(t)$ are monotone, the number of $\lambda \in \Lambda$ such that $\gamma \cap S_\lambda \neq \emptyset$ equals $\|(\Delta x, \Delta y)\|_1 + O(1)$, where $(\Delta x, \Delta y) = v_1 - v_2$ and $\|(\Delta x, \Delta y)\|_1 = |\Delta x| + |\Delta y|$ denotes the L^1 norm on $\mathbb{R}^2$.

Proof. Since $\max(|\Delta x|, |\Delta y|) \gg |v_1 - v_2|$; say $|\Delta x| \geq |\Delta y|$ and thus $|\Delta x| \gg |v_1 - v_2|$. By the intermediate value theorem γ must intersect at least $|\Delta x| - 1$ distinct vertical lines with integer x-coordinates, and these intersections yield a distinct set of λ such that $\gamma \cap S_\lambda \neq \emptyset$. The argument for $|\Delta x| \leq |\Delta y|$ is similar. The argument for the second assertion is similar. □

Corollary 5. *If $k = o(R)$, both λ_j, λ_{j+k} must be contained in a circular sector having arc measure $O(k/R) = o(1)$; in particular, $\arg(\lambda_{j+k}) - \arg(\lambda_j) = o(1)$.*

As for the number of lattice points in quadrants and octants we have $|\Lambda(R, (3\pi/2, 2\pi))| = 2R + O(1)$, and $|\Lambda(R, (3\pi/2, 7\pi/4))| = R + O(1)$.

Further, for angular intervals $I = [\theta_1, \theta_2) \subset (3\pi/2, 2\pi)$ shrinking with R (i.e. $|I| = o(1)$ as $R \to \infty$) we have

$$|\Lambda(R, I)| = R \cdot |I| \cdot (|\cos\theta_1| + |\sin\theta_1|) \cdot (1 + o(1)),$$

provided $|I| \cdot R \to \infty$.

Proof. The two first assertions are immediate consequences of the second assertion of Lemma 4. For the third assertion, we note that both the x and y coordinates are monotone since the curve segment is contained in the fourth quadrant. Since $\Delta x = R \cdot |I| \cos(\theta) \cdot (1 + o(1))$, and similarly $\Delta y = R \cdot |I| \sin(\theta) \cdot (1 + o(1))$ the third assertion follows. □

Given lattice points $\lambda_1, \lambda_2 \in \Lambda$ we define the distance between them as

$$d(\lambda_1, \lambda_{2)}) := \|\lambda_1 - \lambda_2\|_1.$$

For a circular arc C (of large radius and centered at the origin) contained in one of the quadrants, starting at some point in S_{λ_1} and ending at some point in S_{λ_2}, the arc traverses exactly $d(\lambda_1, \lambda_2)$ distinct squares S_λ. In particular, if λ_j, λ_{j+k} (ordered by argument as described above) lie in the same quadrant, we have $d(\lambda_j, \lambda_{j+k}) = k$; conversely if $\lambda_j, \lambda \in \Lambda(R)$ lie in the same quadrant, with $\arg(\lambda) > \arg(\lambda_j)$, and $d(\lambda, \lambda_j) = k$, then $\lambda = \lambda_{j+k}$.

2.2. **Reducing to lattice points in the seventh octant and $k > 0$.** We begin by reducing to the case where angular interval I lies in the seventh octant, so that the slope of the tangent lines lie in $(0, 1)$; this will be convenient later when we subdivide the circle into pieces where the tangent slope lies in intervals given by the Farey dissection of $(0, 1)$.

Given an angular interval $I \subset (0, 2\pi)$, recall that $\Lambda(R, I) := \{\lambda \in \Lambda(R) : \arg(\lambda) \in I\}$ and the corresponding local area correlation is given by $\mathrm{Corr}(R, k, I) = \frac{1}{|\Lambda(R,I)|} \sum_{\lambda_j \in \Lambda(R,I)} (A(\lambda_j) - 1/2)(A(\lambda_{j+k}) - 1/2)$. After suitably subdividing I, we may assume that $I \subset (\ell\pi/4, (\ell+1)\pi/4)$ for some integer ℓ, so that the corresponding lattice points $\{\lambda_j\}$ lie in one of the eight octants (in case $\Lambda(R)$ has points on the coordinate axes their contribution is negligible since there are at most four such.) Further, since $\mathbb{Z}^2$ is invariant under rotation by $\pi/2$, and the reflection in the real axis, we may assume (possibly after changing the sign of k) that I lies in the seventh octant $(3\pi/2, 7\pi/4)$, and consequently that the slope of the tangent lines of $C(R)$, in said octant, lies in $(0, 1)$.

By Corollary 5, both λ_j, λ_{j+k} are contained in some arc of size $\ll k$. Consequently, if $k < 0$, there are $O(k)$ lattice points for which λ_{j+k} lies in the sixth octant and λ_j lies in the seventh octant, and these give a negligible contribution to the local correlation sum as long as I is not too small, in particular if $k/R = o(|I|)$.

Now, if $k < 0$ and λ_j, λ_{j+k} both lie in an interval contained in the seventh octant, we have

$$\sum_{\lambda_j \in \Lambda(R,I)} (A(\lambda_j) - 1/2)(A(\lambda_{j+k}) - 1/2)$$

$$= \sum_{\lambda_j \in \Lambda(R,I')} (A(\lambda_j) - 1/2)(A(\lambda_{j-k}) - 1/2)$$

where $I' = I + O(k/R)$ is some other interval contained in the seventh octant. Provided $k/R = o(|I|)$ this allows us to assume that $k > 0$ without changing the asymptotics of the local correlation sum.

3. A probabilistic model for a family of circles

We introduce a probabilistic model for the intersections of $C(R)$ with S_λ, for λ in the seventh octant of the circle. We will view $C(R)$ as oriented counter clockwise, and we are interested in the distribution of the points where $C(R)$ first enters each S_λ. As we restrict λ to the seventh octant, all such points of first intersection (for R sufficiently large) will be in the lower, or left part of the boundary of S_λ, and we let

$$\partial_L(S) := \{(x,0) : x \in [0,1)\} \cup \{(0,y) : y \in [0,1)\} \subset S$$

denote the L-shaped lower, or left, part of the boundary of S. Similarly we let $\partial_L(S_\lambda)$ denote the lower/left part of the boundary of S_λ; we note that $\cup_{\lambda\in\Lambda}\partial_L(S_\lambda)$ is a disjoint union of $\{(x,y) \in \mathbb{R}^2 : x \in \mathbb{Z} \text{ or } y \in \mathbb{Z}\}$, the set of horizontal and vertical lines with one integer coordinate.

After subtracting by λ, the intersection $C(R) \cap S_\lambda$ is translated to lie in S, and said translate of $C(R)$ is a circle passing through $\partial_L(S)$ at a point $v(\lambda)$, having a tangent line passing through $v(\lambda)$ with slope $\tan(\theta(\lambda))$, for some angle $\theta(\lambda) \in (0, \pi/4)$. Formally, we define $v : \Lambda(R, (3\pi/2, 7\pi/4)) \to \partial_L(S)$ by

$$v(\lambda) := (C(R) \cap \partial_L(S_\lambda)) - \lambda$$

(note that the translation of $\partial_L(S_\lambda)$ by $-\lambda$ equals $\partial_L(S)$, and that $C(R) \cap \partial_L(S_\lambda)$ consists of a single point for R sufficiently large since λ lies in the 7th octant.) We further define $\theta : \Lambda(R, (3\pi/2, 7\pi/4)) \to (0, \pi/4)$ as the angle between the tangent line of $C(R)$, at the point of intersection $C(R) \cap \partial_L(S_\lambda)$, and the horizontal coordinate axis.

We will next define a probability measure on the said family of circles passing through the points of $\partial_L(S)$; later we will show that this measure is a good model for the joint distribution of $(\theta(\lambda), v(\lambda))$ for $\lambda \in \Lambda(R, (3\pi/2, 7\pi/4))$.

For $\theta \in (0, \pi/4)$ define a measure μ_θ on $\partial_L(S)$ by

$$\mu_\theta(J) = \begin{cases} |J| \cdot \sin\theta & \text{if } J \subset \{(x,0) : x \in [0,1)\}, \\ |J| \cdot \cos\theta & \text{if } J \subset \{(0,y), y \in [0,1)\}, \end{cases}$$

where J denotes an interval, of length $|J|$, contained in either the vertical or horizontal part of $\partial_L(S)$. Given a point $v \in \partial_L S$ and an angle $\theta \in (0, \pi/4)$ there is a unique line $L_{v,\theta}$ passing through v having slope $\tan(\theta)$. Further, there is a unique circle $C_{\theta,v} = C_{\theta,v}(R)$, of radius R, passing through v, with tangent line having slope $\tan\theta$, and whose center lies *above* $L_{v,\theta}$.

We define a probability measure μ on the collection of circles

$$\{C_{\theta,v} : v \in \partial_L S,\ \theta \in (0, \pi/4)\}$$

via

$$\mu(I \times J) = \frac{1}{\pi/4} \int_I \mu_\theta(J)\, d\theta$$

for $I \subset (0, \pi/4)$ an interval, and $J \subset \partial_L S$ a horizontal or vertical interval. It will be convenient to use the shorthand

$$d\mu = \begin{cases} \sin\theta\, d\theta\, dx, \\ \cos\theta\, d\theta\, dy, \end{cases} \tag{1}$$

where dx is viewed as a measure on the horizontal part of $\partial_L(S)$ and dy as a measure on the vertical part of $\partial_L(S)$.

3.1. **Correlations via the family of circles.** To study the joint distribution of $A(\lambda_j), A(\lambda_{j+k})$, as j varies, we translate each intersection $S_{\lambda_j} \cap C(R)$ by $-\lambda_j$, and obtain a family of circles $C_{v(\lambda_j),\theta(\lambda_j)}$; each translation maps the pair $(\lambda_j, \lambda_{j+k})$ to $(0, \lambda'_{j,k})$ where $d(\lambda'_{j,k}, 0) = k = \|\lambda'_{j,k}\|_1 + O(1)$, and $\lambda'_{j,k}$ lies in the first octant (as $k > 0$.)

Fix $v \in \partial_L S$. Given a lattice point λ in the first octant such that $d(0,\lambda) = k$ (with k large) the set of $\theta \in (0, \pi/4)$ such that $C_{\theta,v} \cap S_\lambda \neq \emptyset$ consists of an open interval $I = (\theta_1, \theta_2) \subset (0, \pi/4)$, with $|I| = \theta_2 - \theta_1 \asymp 1/k = o(1)$, as k grows.

Further, letting $B_{\theta,v}$ denote the ball having boundary $C_{\theta,v}$, it is straightforward to see that

$$|B_{\theta,v} \cap S| = |B_{v,\theta_1} \cap S| + o(1)$$

if $|\theta - \theta_1| = o(1)$. This will be used to show that the areas inside $B_{\theta,v}$ and the two squares S and S_λ *approximately decouple* as long as the angle variation is $o(1)$, in particular that

$$\int_I |B_{\theta,v} \cap S| \cdot |B_{\theta,v} \cap S_\lambda|\, d\theta = |B_{v,\theta_1} \cap S| \cdot \int_I |B_{\theta,v} \cap S_\lambda|\, d\theta + o(1)$$

for $|I| = o(1)$.

To start, we show that knowledge of θ and v, with sufficient precision and provided we avoid "corner cases", uniquely determines λ such that that $C_{\theta,v} \cap S_\lambda \neq \emptyset$ with $d(\lambda, 0) = k = \|\lambda - v\|_1 + O(1)$, as well as determining the area $|B_{\theta,v} \cap S_\lambda|$ up to a small error. More precisely, that both $C_{\theta,v}$ and $C_{v',\theta'}$ intersect S_λ provided that $|v - v'| = o(1)$, $|\theta - \theta'| = o(1/|\lambda|)$, and that the intersection $C_{\theta,v} \cap S_\lambda$ is close neither to the upper left corner, nor the lower right corner, of S_λ. Furthermore, in this case we also have $|B_{\theta,v} \cap S_\lambda| = |B_{v',\theta'} \cap S_\lambda| + o(1)$.

We begin by showing that the set of angles θ for which $C_{\theta,v}$ intersects S_λ, for $\lambda \in \Lambda$ fixed with $d(0,\lambda) = k$ and k large, depends mildly on the point v.

Lemma 6. *Let $\lambda \in \Lambda$ with $|\lambda| = o(R)$ and $1/|\lambda| = o(1)$, and fix $v \in \partial_L(S)$. Then $I_v(\lambda) := \{\theta : C_{\theta,v} \cap S_\lambda \neq \emptyset\}$ is an interval (α, β) of length $|I_v(\lambda)| \asymp 1/|\lambda|$. Further, if $v' \in \partial_L(S)$ satisfies $|\lambda|/R \leq |v - v'| \leq \delta$, then $I_{v'}(\lambda) = (\alpha + O(\delta/|\lambda|), \beta + O(\delta/|\lambda|))$.*

Proof. Here and in what follows it will be convenient to identify $\mathbb{R}^2$ with $\mathbb{C}$. Let $\theta_1 < \theta_2$ denote angles such that C_{v,θ_1} passes through $\lambda + 1$ and C_{v,θ_2} passes through $\lambda + i$, and let L_1, L_2 denote secant lines passing through $v, \lambda_1 + 1$ respectively $v, \lambda_1 + i$, say having slopes $\tan \theta_1'$ respectively $\tan \theta_2'$. Then $\theta_1' - \theta_1 = \arcsin \frac{|\lambda+1-v|}{2R} = \arcsin \frac{|\lambda|}{2R} + O(1/R)$ and similarly $\theta_2' - \theta_2 = \frac{|\lambda|}{2R} + O(1/R)$. Thus, since $\theta_2' - \theta_1' = \arg(\lambda + i - v) - \arg(\lambda + 1 - v) \asymp 1/|\lambda|$ we find that $\beta - \alpha = \theta_2 - \theta_1 = \theta_2' - \theta_1' + O(1/R) \asymp 1/|\lambda| + O(1/R) \asymp 1/|\lambda|$.

The argument for the second assertion is similar: let α' denote the angle such that $C_{v',\alpha'}$ passes through $\lambda + 1$ and β' the angle such that $C_{v',\alpha'}$ passes through $\lambda + i$. On letting L_1, L_2 denote two secant lines passing through $v, \lambda + 1$ and $v', \lambda + 1$ we find, as before, that $\alpha' - \alpha = \arg(\lambda+1-v) - \arg(\lambda+1-v') + O(1/R) = O\left(|v - v'|/|\lambda| + O(1/R)\right) = O(\delta/|\lambda|)$. We similarly find that $\beta' = \beta + O(\delta/|\lambda|)$. $\square$

Lemma 7. *Let $I \times J$ be a product of intervals such that $C_{\theta,v} \cap S_\lambda \neq \emptyset$ for all $(\theta, v) \in I \times J$, with $|I| = o(1/|\lambda|)$, $|\lambda| = o(R)$, and $|J| = o(1)$, as R grows. Then, if $(\theta', v') \in I \times J$ we have*

$$|B_{\theta,v} \cap S_\lambda| = |B_{v',\theta'} \cap S_\lambda| + o(1)$$

for all $(\theta, v) \in I \times J$, as R grows. Further, if $(\theta(\lambda_i), v(\lambda_i)) \in I \times J$ then $\lambda_{i+k} = \lambda_i + \lambda$.

Proof. Write $\lambda = (x, y)$ and let $\lambda + 1 = (x + 1, y(v, \theta))$ denote the first point of intersection between $C_{\theta,v}$ (starting at v and traversing the circle counter clockwise) and the vertical line L passing through $(x + 1, 0)$. As the family of circles $C_{\theta,v}$ intersects the line transversally, with angle of intersection uniformly bounded away from zero (here we use that $|\lambda| = o(R)$), the function $y(\theta, v)$ is smooth, with $\partial y/\partial v = O(1)$ and $\partial y/\partial \theta = O(|\lambda|)$. Under our assumptions it follows that $y(\theta, v) = y(\theta', v') + o(1)$ for all $(\theta, v) \in I \times J$. Noting that the area $|B_{\theta,v} \cap S_\lambda|$ is a piecewise smooth function of y, the result follows. $\square$

We next control the contribution of "corner case", i.e., the contribution from $(\theta, v) \in C_{\theta,v} \cap S_\lambda \neq \emptyset$ for *some* $(\theta, v) \in I \times J$ but not for *all* $(\theta, v) \in I \times J$.

Lemma 8. *Fix $\lambda \in \Lambda$, with $d(0,\lambda) \asymp |\lambda|$ large and $\arg(\lambda) \in (0, \pi/4)$. Let $J \subset \partial_L(S)$ be a horizontal or vertical interval of length one, and let I_λ be a minimal (open) interval such that the set of (θ, v) for which $C_{\theta,v} \cap S_\lambda \neq \emptyset$ is contained in $I_\lambda \times J$. Then $|I_\lambda| \asymp 1/|\lambda|$, and we may choose integers $M = M(R)$, slowly growing with R such that if we divide I_λ into M parts and J into M parts, we cover the set $\{(\theta, v) : C_{\theta,v} \cap S_\lambda \neq \emptyset\}$ with $\asymp M^2$ "internal boxes" B_{ij}, $1 \leq i, j \leq M$, having side lengths $\asymp 1/(|\lambda|M)$ and $1/M$, and the additional property that $C_{\theta,v} \cap S_\lambda \neq \emptyset$ for all $(\theta, v) \in B_{ij}$, and $O(M)$ "boundary" boxes B_{ij} with the property that $C_{\theta,v} \cap S_\lambda = \emptyset$ for some $(\theta, v) \in B_{ij}$, and $C_{\theta',v'} \cap S_\lambda \neq \emptyset$ for some other $(\theta', v') \in B_{ij}$.*

Proof. Given $v \in \partial_L(S)$ the set of θ such that $C_{\theta,v} \cap S_\lambda \neq \emptyset$ is an open interval $(\theta_1(v), \theta_2(v))$. Let $L_1(v), L_2(v)$ denote the two lines passing through $v, \lambda + 1$ and $v, \lambda + i$; we note that $C_{v,\theta_1(v)} \cap L_1(v) = \{v, \lambda + 1\}$ and $C_{v,\theta_2(v)} \cap L_2(v) = \{v, \lambda + i\}$. Let $\phi_1(v) := \arg(\lambda + 1 - v)$ denote the angle between L_1 and the horizontal axis, and let $\phi_2(v) := \arg(\lambda + i - v)$ denote the angle between L_2 and the horizontal axis. We then find, as $\arg(\lambda) \in (0, \pi/2)$, that

$$\begin{aligned} (2) \quad \phi_2(v) - \phi_1(v) = \arg((\lambda + i - v)/(\lambda + 1 - v)) = \\ \arg((1 + i/(\lambda - v))/(1 + 1/(\lambda - v))) = \\ \arg(1 + (i - 1)/(\lambda - v)) + O(1/|\lambda|^2) \asymp 1/|\lambda|. \end{aligned}$$

Similarly, if $v' \in \partial_L(S)$, we find that

$$(3) \qquad \phi_j(v) - \phi_j(v') \ll |v - v'|/|\lambda| + O(1/|\lambda|^2)$$

for $j = 1, 2$.

Further, since $\phi_1(v) - \theta_1(v)$ is the angle between the tangent line of C_{v,θ_1} at v and a chord on C_{v,θ_1}, of length $|\lambda + 1 - v|$, we find that $\phi_1(v) - \theta_1(v) = \arcsin(|\lambda + 1 - v|/(2R)) = \arcsin(|\lambda|/(2R)) + O(1/R)$, uniformly for $v \in \partial_L(S)$. Similarly, $\phi_2(v) - \theta_2(v) = \arcsin(|\lambda|/(2R)) + O(1/R)$ also holds uniformly in v. Consequently, we have

$$\theta_j(v) - \theta_j(v') \ll |v - v'|/|\lambda| + O\left(1/|\lambda|^2 + 1/R\right).$$

In particular, we find that $\theta_j(v) - \theta_j(v') \ll 1/|\lambda|$ which, together with $\phi_2(v) - \phi_1(v) \asymp 1/|\lambda|$, implies that the minimal interval I is of size $\asymp 1/|\lambda|$. Similarly, subdividing $I_\lambda \times J$ into $\asymp M^2$ boxes B_{ij} having side lengths $\asymp 1/(M|\lambda|)$ and $1/M$ we find that, for each i there, are $O(1)$ values of j for which B_{ij} is a boundary box, for a total of $O(M)$ boundary boxes, and $\asymp M^2$ internal boxes. $\square$

3.2. **Second intersection area averages.** To show that the average of $|S_\lambda \cap B_{\theta,v}|$, as θ ranges over elements in the interval $I = (\theta_1, \theta_2) = \{\theta : C_{\theta,v} \cap S_\lambda \neq \emptyset\}$, equals $1/2 + o(1)$ (for $R, |\lambda|$ large) we approximate $C_{\theta,v}$ by a family of lines passing through v, and in turn approximate these lines by a family of parallel lines.

More precisely, chose n to be a unit vector orthogonal to the vector $\lambda = \lambda - 0$, and let L be the line passing through λ in the direction given by n. As θ ranges over angles in I, the intersection point $w = w(\theta) = L \cap C_{\theta,v}$ can be written as $w = \lambda + t \cdot n$ for $t = t(\theta)$ ranging over some interval T. Moreover, since k is growing, we can write (cf. (1)) $dt = (1 + o(1)) \cdot \sin\theta_1 \, d\theta$ for x fixed (or $dt = (1 + o(1)) \cdot \cos\theta_1 \, d\theta$ for y fixed). In other words, the pushforward of the measure μ, for $v = (x, 0)$ or $v = (0, y)$ fixed, to the interval T is, up to negligible error, the uniform measure (up to a scalar depending on θ)

Let L_t denote the line through the point $\lambda + t \cdot n$, in the direction of λ, and let H_{L_t} denote the half-plane bounded by L_t and containing the center of $C_{\theta,v}$. We then have $\arg(w - v) = \theta + O(k/R) = \theta + o(1)$, since the curvature of $C_{\theta,v}$ equals $1/R$ and $|v - \lambda| \asymp k = o(R)$, and thus $|B_{\theta,v} \cap S_\lambda| = |H_{L_{t(\theta)}} \cap S_\lambda| + o(1)$.

Moreover, if we let $T' = \{t : L_t \cap S_\lambda \neq \emptyset\} = (t_1, t_2)$, we have $t_i = t_{\theta_i} + o(1)$ for $i = 1, 2$. In particular, the average of $|H_{L_{t(\theta)}} \cap S_\lambda|$ for $\theta \in (\theta_1, \theta_2)$ is, up to an $o(1)$-error, the same as the average $|H_{L_t} \cap S_\lambda|$ for $t \in T'$. We next show that the average of the area contained in H_t and S_λ, as $t \in T'$, equals $1/2$.

Lemma 9. *Given a unit vector $u \in \mathbb{R}^2$ (not parallel to the axes), define a family of parallel lines $\{L_t\}_{t\in\mathbb{R}}$ defined by $L_t \perp u$ and $t \cdot u \in L_t$, and a collection of half planes H_t having L_t as the boundary, and u pointing into H_t. Given a square S_λ the set $T := \{t : S_\lambda \cap L_t \neq \emptyset\}$ is a finite open interval, and we have*

$$\int_T |H_t \cap S_\lambda| \, dt = |T|/2.$$

Proof. After applying a translation composed with a rotation we may assume that the collection of lines $\{L_t\}$ are parallel with the x-axis, and where S_λ is replaced by a square S', having center of mass at the origin (but sides not necessarily parallel with the coordinate axes), and $T = [-\tau, \tau]$ for some $\tau \in [1/2, 1/\sqrt{2}]$. Letting $A(t)$ denote the area above L_t we find, by reflecting in the x-axis, that $A_1(-t) = 1 - A_1(t)$. Thus

$$\int_{-\tau}^{\tau} A_1(t) \, dt = \int_0^\tau A_1(t) \, dt + \int_0^\tau (1 - A_1(t)) \, dt = \tau = \frac{|T|}{2}.$$

□

4. From lattice points to the continuum model

4.1. **Circle arc approximation via rational slope lines.** We begin by studying the distribution of the intersections $(C(R)\cap\partial_L(S_\lambda))-\lambda$ for $\lambda\in C(R,I)$ and I a small angular interval. To do this, we approximate the circular arc, first by the tangent line, which is then approximated by a line with rational slope a certain Farey fraction. Given $x,y\in\mathbb{R}$, let $D(x,y)$ denote the distance between $x \mod 1$ and $y \mod 1$, i.e., $D(x,y):=\min_{k\in\mathbb{Z}}|x-y-k|$.

Lemma 10. *Let $f\in C^2([-T,T])$ be in increasing function such that $f(0)=\alpha$ and $\max_{t:|t|\le T}(|f(T)-(\alpha+\beta t)|)\le\epsilon$ for some $\beta>0$. For integer $k\in[-T,T]$, let $y_k=f(k)$, and put $\tilde{y}_k=\beta k$. Similarly, for integer $l\in[f(-T),f(T)]$, let x_l denote the unique solution to $f(x)=l$, and similarly let $\tilde{x}_l$ denote the unique solution to $\alpha+\beta x=l$. We then have $D(y_k,\tilde{y}_k)<\epsilon$ for all $k\in[-T,T]\cap\mathbb{Z}$. Similarly, we have $D(x_l,\tilde{x}_l)\le\epsilon/\beta$ for $l\in[f(-T),f(T)]\cap\mathbb{Z}$.*

Proof. Since $D(x,y)\le|x-y|$ the first assertion follows from the fact that $\max_{t\in[-T,T]}|f(t)-(\alpha+\beta t)|\le\epsilon$. As for the second assertion, we note that $f(x_l)=l$ and $|f(x_l)-(\alpha+\beta x_l)|\le\epsilon$ implies that $\alpha+\beta x=l$ has a solution within distance at most ϵ/β from x_l.

□

Definition 1. *Let $N,M>0$ be integers. We say that a sequence $\{x_n \mod 1\}_{n\le N}$ is ϵ-equidistributed at scale M if for all intervals $J\subset[0,1]$ satisfying $|J|>1/M$, we have $|\{n\le N: x_n\in J\}|=N\cdot|J|\cdot(1+\theta)$ for some θ with $|\theta|<\epsilon$.*

For simplicity, we say that such a sequence is “approximately equidistributed”.

After locally approximating $C(R)$ by lines whose slope β is given by Farey fractions, we will be concerned with the intersection of lines of the form $y=\alpha+\beta x$ for $\beta=p/q$ a rational number (we tacitly assume $(p,q)=1$) whose height $\max(p,q)$ grows, with horizontal and vertical lines with at least one integer coordinate.

Lemma 11. *Let $0<p<q$ be coprime integers, and fix $\alpha\in[0,1]$. If $M\le q/\log q$ and $N\ge q\log q$ then $\{\alpha+\frac{p}{q}\cdot n \mod 1\}_{n\le N}$ is ϵ-equidistributed at scale M for $\epsilon=O(1/\log q)$.*

Similarly, for the x-coordinates, $\mod 1$, of the intersections with horizontal lines we obtain points x_m such that $\alpha+\frac{p}{q}x_m=m\in\mathbb{Z}$, i.e.,

$x_m = \frac{q}{p}m + \alpha'$ *for some* $\alpha' \in \mathbb{R}$ *and thus approximate equidistribution holds with* $\epsilon = O(1/\log p)$ *if* $M \leq p/\log p$ *and* $N \geq p \log p$.

Proof. The sequence

$$x_n := \left\{\alpha + \frac{p}{q}n\right\}$$

is periodic modulo 1 with period q. Since $(p, q) = 1$, the set of values attained over one period is exactly a translate of the rational grid $q^{-1}\mathbb{Z}/\mathbb{Z}$. Consequently, for any interval $J \subset [0, 1]$,

$$\#\{1 \leq n \leq q : x_n \in J\} = q|J| + O(1),$$

uniformly in J and α.

Write $N = aq + r$ with $0 \leq r < q$. Then

$$\#\{n \leq N : x_n \in J\} = a\,\#\{1 \leq n \leq q : x_n \in J\} + O(q|J|),$$

since the final incomplete block contributes at most $O(q|J|)$ points. Using the previous estimate and $a = N/q + O(1)$, we obtain

$$\#\{n \leq N : x_n \in J\} = N|J| + O\left(\frac{N}{q} + q|J|\right).$$

Now assume $|J| > 1/M$. From the hypothesis $M \leq q/\log q$ we obtain $q|J| > \log q$. Also $N \geq q \log q$, hence

$$\frac{1}{q|J|} \ll \frac{1}{\log q}, \qquad \frac{q}{N} \ll \frac{1}{\log q}.$$

Dividing the previous estimate by $N|J|$ therefore gives

$$\#\{n \leq N : x_n \in J\} = N|J|\Big(1 + O\big(\tfrac{1}{\log q}\big)\Big),$$

uniformly for all intervals $J \subset [0, 1]$ with $|J| > 1/M$ which is the claimed result. □

4.2. **Rational slope lines and Farey fractions.** For $Q > 0$ an integer let $F_Q := \{p/q : 0 \leq p \leq q \leq Q, (p, q) = 1\}$ denote the set of Farey fractions of height at most Q.

Given $p/q \in F_Q \setminus \{0, 1\}$, define a half open interval

$$\begin{aligned} I_{p,q,Q} :&= [(p + p')/(q + q'), (p + p'')/(q + q'')) \\ &= [p/q - 1/(q(q + q')), p/q + 1/(q(q + q''))) \end{aligned} \tag{4}$$

where $p'/q' < p/q < p''/q''$ are consecutive elements in F_Q. The collection of intervals $\{I_{p,q,Q}\}_{p/q \in F_Q \setminus \{0,1\}}$ is the *Farey dissection*, and gives a disjoint union of $[1/(Q + 1), Q/(Q + 1))$ (cf. [3, §3.8]). Since the mediant $(p + p')/(q + q')$ is not in F_Q, we must have $q + q' > Q$ and thus $|I_{p,q,Q}| \asymp 1/(qQ)$; in particular $|I_{p,q,Q}| \gg 1/Q^2$.

We begin by bounding the number of lattice points associated with parts of the circle where the tangent line slope is well approximated by rationals having low height. For comparison, we remark that the number of $\lambda \in \Lambda(R,(3\pi/2,7\pi/4))$ such that $\tan(\arg(\lambda)-3\pi/2) \in I$ is $\asymp R|I|$ provided that $R|I| \gg 1$.

Lemma 12. *Let $I \subset [0,1]$ be a closed interval. The number of $\lambda \in \Lambda(R,(3\pi/2,7\pi/4))$ such that $\tan(\arg(\lambda)-3\pi/2) \in I_{p,q,Q}$, either for some integers $0<p<q \leq Q/\log Q$, or $0<p<q/\log q$ and $q \in [Q/\log Q, Q]$, and additionally that $p/q \in I$, is $o(R|I|)$ provided that $Q \leq \sqrt{R}/\log R$ and $|I| \geq (\log\log Q)/\log Q$.*

Proof. First note that the number of $\lambda \in \Lambda(R,(3\pi/2,7\pi/4))$ such that $\tan(\arg(\lambda)) \in I_{p,q,Q}$ is $\ll R/(qQ)$ provided that $R/(qQ) \gg 1$, which holds given that $q \leq Q \leq \sqrt{R}/\log R$. Further, the number of integer $p \leq q$ such that $p/q \in I$ is $q|I| + O(1)$.

Thus, the number of λ of the first type (i.e., for which $p < q \leq Q/\log Q$) is

$$\ll \sum_{q \leq Q/\log Q} \sum_{p \leq q, p/q \in I} (\frac{R}{qQ} + O(1)) \ll \sum_{q \leq Q/\log Q} (\frac{R}{qQ} + O(1))(q|I| + O(1))$$
$$\ll \frac{R|I|}{\log Q} + \frac{Q^2}{\log^2 Q}|I| + R\frac{\log Q}{Q} + \frac{Q}{\log Q} = o(|I|R)$$

since $\frac{\log Q}{Q} = o(|I|)$ and $\frac{Q}{\log Q} = R\frac{Q}{R\log Q} \ll R\frac{Q}{Q^2\log^2 R \log Q} = Ro(|I|)$.

To bound the number of λ of the second type (i.e., for which $p < q/\log q$ and $q \in [Q/\log Q, Q]$) we first note that the number of $p < q/\log q$ for which $p/q \in I$ for q fixed is

$$\ll \min(q/\log q, q|I| + O(1)) = q/\log q$$

for Q sufficiently large since $|I| \geq \log\log(Q)/\log(Q)$ (note that $\log q \asymp \log Q$ for $q \in [Q/\log Q, Q]$ and Q large.) Thus the number of the second type of λ is

$$\ll \sum_{q \in [Q/\log Q, Q]} \frac{q}{\log q}\left(\frac{R}{qQ} + O(1)\right) \ll \frac{R}{\log Q} + \frac{Q^2}{\log Q} = o(R|I|)$$

since $Q \leq \sqrt{R}/\log R$, and $|I| \geq (\log\log Q)/\log Q$.

□

4.3. **μ-equidistribution in very short angular intervals.** Given Q, define $F_{Q,\mathrm{Good}} := \{p/q \in F_Q : q \in [Q/\log Q, Q], p \in [q/\log q, q]\}$. We may then prove μ-equidistribution of $(\theta(\lambda), v(\lambda)$ for λ ranging over sets such that $(\tan(\theta(\lambda)), v(\lambda)) \in I \times J \subset I_{p,q,Q} \times J$, for I, J of very small sizes (i.e., $|I| \asymp (\log R)^2/R$) and $J \subset \partial_L(S)$, with $|J|$ allowed to slowly shrink.)

Proposition 13. *Let $J \subset \partial_L S$ be a vertical or horizontal interval, and let $I \subset I_{p,q,Q}$ for $p/q \in F_{Q,Good}$. Assume further that $|J| > \log^4 Q/Q$, for $Q = Q(R) \leq \log R$ tending to infinity with R, and that $|I| \asymp \log^2 R/R$. Then the number of $\lambda \in C(R, (3\pi/2, 7\pi/4))$ such that $(\theta(\lambda), v(\lambda)) \in I \times J$ is*

$$R \cdot \mu(I \times J) \cdot (1 + o(1))$$

as $R \to \infty$.

Proof. We begin by noting that Lemma 10 (we subdivide I into intervals of length $(Q \log Q)/R$ and can then take $T \asymp Q \log Q$, $\epsilon \asymp (Q \log Q)^2/R$, and $\beta \gg p/q \gg 1/\log Q$) allows us to approximate the circular arc by a tangent line with slope $\beta \in I_{p,q,Q}$, inducing an error of size $O(Q^3/R)$ in terms of the intersection with lines having at least one integer coordinate.

This tangent line, again using Lemma 10 (with $T = Q \log Q$ as before, noting that $p/q \gg 1/\log Q$ and $|\beta - p/q| \ll \log Q/Q^2$ (since $p/q \in F_{Q,good}$) we can in turn approximate by a line with slope $p/q \in F_{Q,\mathrm{Good}}$, inducing an error of size $O(\log^3 Q/Q)$ in terms of intersections with lines having at least one integer coordinate. By Lemma 11 (with $N = Q \log Q$) we obtain that the boundary intersections points $\{v(\lambda)\}_{\lambda \in \Lambda(R,I)}$ are μ-equidistributed on $\partial_L(S)$ at scale slightly worse than $Q/\log^3 Q$; say $Q/\log^4 Q$. In particular, note that $|I| = o(1)$ implies that the density function of the projection of μ onto the v-component is, up to negligible error, constant as θ ranges over the interval I. $\square$

5. Proof of Theorem 1

The following main technical result allows us show vanishing area correlations for λ_i ranging over lattice points such that $\lambda_{i+k} - \lambda_k = \lambda$ is fixed (this corresponds to very short angular intervals in case k is large) *provided* that $\tan(\arg(\lambda))$ is not well approximated by rationals of small height.

Proposition 14. *Given $\lambda \in \Lambda$ in the first octant with $d(\lambda, 0) = k$ there exist an interval $I(\lambda)$ of length $|I(\lambda)| \asymp 1/k$ such that $\lambda_{i+k} - \lambda_i = \lambda$ implies that $\theta(\lambda_i) \in I(\lambda)$. If $\{\tan(\theta) : \theta \in I(\lambda)\}$ is contained in some $I_{p,q,Q}$ for $p/q \in F_{Q,Good}$, and $Q = Q(R) \leq \min(\log R, \sqrt{k}/\log k)$, then*

$$\sum_{\lambda_i : \lambda_{i+k} - \lambda_i = \lambda} (|B(R) \cap S_{\lambda_i}| - 1/2)(|B(R) \cap S_{\lambda_{i+k}}| - 1/2) = o(N_\lambda) \tag{5}$$

where $N_\lambda = |\{\lambda_i : \lambda_{i+k} - \lambda_i = \lambda\}| \asymp R/k$.

Proof. The interval $I(\lambda) := \{\theta : C_{\theta,v} \cap S_\lambda \neq \emptyset \text{ for some } v \in \partial_L(S)\}$ has the property that $\lambda_{i+k} - \lambda_i = \lambda$ implies that $\theta(\lambda_i) \in I(\lambda)$. and by Lemma 6 we have $|I(\lambda)| \asymp 1/k$.

Let $M = M(R,k)$ denote integers slowly growing with R, k, and let $J \subset \partial_L(S)$ denote an half open interval, either vertical or horizontal, of length $1/M$. Covering $\partial_L(S)$ by $2M$ such intervals (i.e., M horizontal and M vertical ones), we claim that it suffices to show that the localized second intersection area sums are small in the sense that

$$\sum_{\lambda_i:\lambda_{i+k}-\lambda_i=\lambda, v(\lambda_i)\in J} (|B(R) \cap S_{\lambda_{i+k}}| - 1/2) = o(N_\lambda/M). \tag{6}$$

To justify the claim, first note that the variation of the first intersection areas $|B(R) \cap S_{\lambda_i}|$, as λ_i ranges over elements such that $v(\lambda_i) \in J$ and $\theta(\lambda_i) \in I(\lambda)$, is $o(1)$. (In fact, this holds as long as $v(\lambda_i)$ and $\theta(\lambda_i)$ varies in intervals of length $o(1)$.) Summing the contribution from the $2M$ such intervals J we arrive at an error term of size $o(N_\lambda)$ in (5).

To show that (6) holds we argue as follows: let $I(\lambda, J) := \{\theta : C_{\theta,v} \cap S_\lambda \neq \emptyset \ \forall v \in J\}$; Lemma 8 then gives $|I(\lambda, J)| \asymp 1/k$. Further, again by Lemma 8 we may cover $I(\lambda, J)$ by M disjoint (half-open) same sized intervals $\{I_j\}_{j\le M}$, with $|I_j| \asymp 1/(kM)$ and where all but $O(1)$ of these M intervals are internal in the sense that $C_{\theta,v} \cap S_\lambda \neq \emptyset$ for **all** $(\theta, v) \in I_j \times J$. In particular, $\lambda_{i+k} - \lambda_i = \lambda$ holds if $(\theta(\lambda_i), v(\lambda_i)) \in I_j \times J$ and I_j is internal.

Since there are $O(1)$ non-internal intervals I_j, each of length $O(1/(kM))$, $|J| = 1/M$, and μ-equidistribution holds at scale $I_j \times J$, the corresponding contribution to (6) is $O(R/(kM^2)) = o(N_\lambda/M)$. Further, again since μ-equidistribution holds at scale $I_j \times J$, we find that for the internal intervals I_j, we have

$$\sum_{\lambda_i:\lambda_{i+k}-\lambda_i=\lambda,(\theta(\lambda_i,v(\lambda_i))\in I_j\times J} (|B(R) \cap S_{\lambda_{i+k}}| - 1/2) =$$
$$R(1 + o(1)) \int_{I_j\times J} (|C_{\theta,v} \cap S_\lambda| - 1/2) d\mu(\theta, v)$$

Summing over all j we find that

$$\sum_{\lambda_i:\lambda_{i+k}-\lambda_i=\lambda,\, \theta(\lambda_i)\in J} (|B(R) \cap S_{\lambda_{i+k}}| - 1/2) =$$
$$R(1 + o(1)) \int_{I(\lambda,J)\times J} (|C_{\theta,v} \cap S_\lambda| - 1/2) d\mu(\theta, v)$$

since the contribution from non internal I_j is negligible, in the sum as well as in the integral. Moreover, with $X\Delta Y$ denoting the symmetric difference between two sets X, Y, we have

$$\mu((I(\lambda, J) \times J)\Delta\{(v,\theta) : v \in J, C_{v,\theta} \cap S_\lambda \neq \emptyset\}) = o(\mu(I(\lambda, J) \times J)).$$

and thus

$$\begin{aligned}\int_{I(\lambda,J)\times J} &(|C_{\theta,v} \cap S_\lambda| - 1/2)d\mu(\theta, v) = \\ &(1 + o(1)) \int_{\{(v,\theta):v\in J,C_{v,\theta}\cap S_\lambda\neq\emptyset\}} (|C_{\theta,v} \cap S_\lambda| - 1/2)d\mu(\theta, v)\end{aligned} \tag{7}$$

which, in case J is a horizontal interval, equals

$$(1+o(1)) \int_{\{(\theta,x):C_{\theta,(x,0)}\cap S_\lambda\neq\emptyset,(x,0)\in J\}} (|C_{\theta,(x,0)}\cap S_\lambda|-1/2) \sin(\theta)\, d\theta\, dx. \tag{8}$$

Since the variation of $\sin(\theta)$ is $o(1)$ for $\theta \in I(\lambda, J)$, we obtain, by the discussion in Section 3.2, that for x fixed,

$$\begin{aligned}\int_{\{\theta:C_{\theta,(x,0)}\cap S_\lambda\neq\emptyset\}} &(|C_{\theta,(x,0)} \cap S_\lambda| - 1/2)d\theta \\ &= o\left(|\{\theta : C_{\theta,(x,0)} \cap S_\lambda \neq \emptyset\}|\right) = o(1/k)\end{aligned}$$

and thus (7) is $o(1/kM)$ and consequently (6) is $o(R/(kM)) = o(N_\lambda/M)$. The argument for J a vertical interval is similar.

□

Corollary 15. *Let $I \subset (3\pi/2, 7\pi/4)$ be an interval such that $\{\tan(\theta - 3\pi/2) : \theta \in I\} = I_{p,q,Q}$ for $p/q \in F_{Q,good}$. Provided $1/k = o(1/(qQ))$ we have*

$$\sum_{\lambda_i:\arg(\lambda_i)\in I} (|B(R) \cap S_{\lambda_i}| - 1/2)(|B(R) \cap S_{\lambda_{i+k}}| - 1/2) = o(R|I|)$$

Proof. Since $|I| \asymp |I_{p,q,Q}| \asymp 1/(qQ)$ we may cover I by intervals $I(\lambda)$, for λ ranging over some subset of $\{\lambda' \in \Lambda : d(0, \lambda') = k\}$. Further, since $|I_\lambda| \asymp 1/k$, the number of such intervals grows with R, with all but $O(1)$ them being contained in I. The result now follows from Proposition 14 by summing over the relevant set of λ's on noting that the contribution from intervals $I(\lambda)$ not contained in I is negligible (since there are only $O(1)$ such)

□

By taking longer intervals I we can allow intervals for which the tangent line is well approximated by rationals of low height.

Corollary 16. *Let $I \subset (3\pi/2, 7\pi/4)$ be an interval such that $|I| > (\log\log Q)/\log Q$, with $Q = \min(\log R, \sqrt{k}/\log k)$. Then*

$$\sum_{\lambda_i : \arg(\lambda_i) \in I} (|B(R) \cap S_{\lambda_i}| - 1/2)(|B(R) \cap S_{\lambda_{i+k}}| - 1/2) = o(R|I|)$$

Proof. Since we assume that $|I| > (\log\log Q)/\log Q$, Lemma 12 implies that the contribution from λ_i such that $\tan(\arg(\lambda_i)) \in I_{p,q,Q}$ for $p/q \notin F_{Q,good}$ is $o(R|I|)$.

Since $1/k = o(1/Q^2)$ the result follows by covering the remaining part of I by intervals $I'_{p,q,Q} = \{\theta : \tan(\theta) \in I_{p,q,Q}\}$ with $p/q \in F_{Q,good}$, and applying Corollary 15.

□

Since the discussion in Section 2.2 allows us to reduce to intervals contained in the seventh octant, Theorem 1 is now an immediate consequence of the previous corollary.

6. Local area mean

For an interval $I \subset (0, 2\pi)$ we define the local area mean as

$$\mathrm{Mean}(R, I) := \frac{1}{|\Lambda(R, I)|} \sum_{\lambda \in \Lambda(R,I)} A(\lambda);$$

recall that $\Lambda(R, I) := \{\lambda \in \Lambda(R) : \arg(\lambda) \in I\}$. The following two results give an asymptotic for the local mean in two ranges. The first is uniform in $\arg(\lambda)$, at the cost of a larger interval, while the second is based on the Diophantine type of $\tan(\arg(\lambda))$ and allows taking much smaller intervals "in generic position".

6.1. Uniform local mean.

We first prove the uniform result.

Proposition 17. *If*

$$|I| > \frac{(\log R)^8}{\sqrt{R}},$$

then, as $R \to \infty$,

$$\mathrm{Mean}(R, I) = \frac{1}{2} + o(1).$$

We remark that the interval length assumption is essentially sharp: near the lowest part of the circle we must traverse more than $R^{\frac{1}{2}}$ squares to obtain equidistribution of the hitting points on the left side of the fundamental domain. We start with a variation of Lemma 12 for large Q; to do so we take $p \in [\log^4 q, q]$ in our definition of "good" Farey fractions in order to control the contribution from small height intervals $I_{p,q,Q}$.

Lemma 18. *Let $I \subset [0,1]$ be a closed interval. The number of $\lambda \in \Lambda(R,(3\pi/2,7\pi/4))$ such that $\tan(\arg(\lambda)-3\pi/2) \in I_{p,q,Q}$, either for some integers*

$$0 < p < q \leq Q/\log Q,$$

or

$$0 < p < (\log q)^4, \quad q \in [Q/\log Q, Q],$$

and additionally that $p/q \in I$, is $o(R|I|)$ provided that

$$Q = \frac{\sqrt{R}}{(\log R)^3}$$

and

$$|I| > \frac{(\log R)^8}{\sqrt{R}}.$$

Proof. The proof is identical to Lemma 12, with the modified range for p. □

We next allow for much smaller p in the definition of $F_{Q,\text{Good}}$ and prove a variation of Proposition 13. Given Q, define

$$\widehat{F}_{Q,\text{Good}} := \{p/q \in F_Q : q \in [Q/\log Q, Q], p \in [(\log q)^4, q]\}.$$

The main differences compared to Proposition 13 is that we need to ensure that I contains some slope β such that $\beta \in I_{p,q,Q}$ for some $\frac{p}{q} \in F_{Q,\text{Good}}$.

Proposition 19. *Let $J \subset \partial_L S$ be a vertical or horizontal interval, and assume that*

$$\exists\, \beta \in I_{p,q,Q} \cap I \quad \textit{for some} \quad \frac{p}{q} \in \widehat{F}_{Q,Good}.$$

Assume further

$$|J| > \frac{1}{\log\log R},$$

$$Q = Q(R) = \frac{\sqrt{R}}{(\log R)^3},$$

and

$$|I| = \frac{1}{\sqrt{R}\log R}.$$

Then the number of $\lambda \in \Lambda(R,(3\pi/2,7\pi/4))$ such that $(\theta(\lambda), v(\lambda)) \in I \times J$ is

$$R \cdot \mu(I \times J) + o(R|I||J|).$$

as $R \to \infty$. In fact, the proof gives the stronger asymptotic

$$R \cdot \mu(I \times J)(1 + o(1))$$

if J is a vertical interval, or if $I \subset [1/(\log\log R)^2, \pi/4]$.

Proof. Here we make a cutoff at $\theta(\lambda) = 1/(\log\log R)^2$ and treat the part with $\theta(\lambda)$ small differently. First, for $\theta(\lambda) > 1/(\log\log R)^2$ we argue as in the proof of Proposition 13. In the smaller range, we distinguish between the lower and left side intersections. The number of lower side intersections is $\ll R \cdot |I|/(\log\log R)^2 = o(R|I||J|)$. For the left side intersections, we again argue as in the proof of Proposition 13. $\square$

Proof of Proposition 17. By the reduction in Section 2.2 we may assume

$$I \subset (3\pi/2, 7\pi/4), \qquad \widehat{I} := I - \frac{3\pi}{2} \subset (0, \pi/4).$$

Let $Q = \sqrt{R}/(\log R)^3$. By Lemma 18, the number of $\lambda \in \Lambda(R, I)$ such that $\tan\bigl(\arg(\lambda) - \frac{3\pi}{2}\bigr) \in I_{p,q,Q}$ for some $p/q \notin \widehat{F}_{Q,\mathrm{Good}}$ is $o(R|I|)$. Since $A(\lambda) \leq 1$, their contribution is negligible. Partition $\widehat{I}$ into disjoint intervals $\widehat{I}_\nu$ such that

$$|\widehat{I}_\nu| = \frac{1}{\sqrt{R}\log R},$$

up to $O(1)$ endpoint intervals. The total contribution from the endpoints is negligible. If $\widehat{I}_\nu$ contains no good slope, which means

$$\forall\, \frac{p}{q} \in \widehat{F}_{Q,\mathrm{Good}}\ \nexists\, \beta \in I_{p,q,Q} \cap \widehat{I}_\nu,$$

then its contribution is negligible by Lemma 18 . Thus it suffices to consider those intervals $\widehat{I}_\nu$ containing some good slope, i.e., that

$$\exists\, \beta \in I_{p,q,Q} \cap \widehat{I}_\nu \quad \text{for some} \quad \frac{p}{q} \in \widehat{F}_{Q,\mathrm{Good}}.$$

Fix such an interval $\widehat{I}_\nu$. By Proposition 19, for any interval $K \subset \partial_L(S)$ with

$$|K| > \frac{1}{\log\log R},$$

we have

$$\#\{\lambda : (\theta(\lambda), v(\lambda)) \in \widehat{I}_\nu \times K\} = R\,\mu(\widehat{I}_\nu \times K) + o(R|\widehat{I}_\nu||K|).$$

Summing over a partition of $\partial_L(S)$ into such intervals K, we find that

$$\sum_{\substack{\lambda\in\Lambda(R,I)\\ \theta(\lambda)\in\widehat{I}_\nu}} A(\lambda) = R\int_{\widehat{I}_\nu\times\partial_L(S)} |B_{\theta,v}\cap S|\,d\mu + o(R|\widehat{I}_\nu|),$$

and similarly for the normalization factor

$$\sum_{\substack{\lambda\in\Lambda(R,I)\\ \theta(\lambda)\in\widehat{I}_\nu}} 1 = R\int_{\widehat{I}_\nu\times\partial_L(S)} d\mu + o(R|\widehat{I}_\nu|).$$

By Lemma 9,

$$\int_{\widehat{I}_\nu\times\partial_L(S)} |B_{\theta,v}\cap S|\, d\mu = \frac{1}{2}\int_{\widehat{I}_\nu\times\partial_L(S)} 1\, d\mu.$$

Hence

$$\sum_{\substack{\lambda\in\Lambda(R,I)\\ \theta(\lambda)\in\widehat{I}_\nu}} A(\lambda) = \frac{1}{2}\#\{\lambda\in\Lambda(R,I) : \theta(\lambda)\in\widehat{I}_\nu\} + o(R|\widehat{I}_\nu|).$$

Summing over all good intervals and adding back the negligible bad and endpoint contributions, we find that

$$\sum_{\lambda\in\Lambda(R,I)} A(\lambda) = \frac{1}{2}|\Lambda(R,I)| + o(R|I|).$$

Since $|\Lambda(R,I)|\asymp R|I|$, the desired result follows. □

6.2. **Local mean in good Farey intervals.** We now prove a local mean result based on the Diophantine type of $\tan(\arg(\lambda))$. As in Section 2.2, we focus on the seventh octant.

Proposition 20. *Let*

$$I = I(R)\subset(3\pi/2, 7\pi/4)$$

be an interval such that

$$\tan\left(I-\frac{3\pi}{2}\right)\subset I_{p,q,Q}$$

for some $p/q\in F_{Q,\mathrm{Good}}$, with $Q=Q(R)\to\infty$, $Q\le\log R$, and

$$|I|\asymp\frac{\log^2 R}{R}.$$

Then

$$\mathrm{Mean}(R,I) = \frac{1}{2}+o(1).$$

Proof. The result follows, as in the proof of Proposition 17, using Lemma 12, Proposition 13 and Lemma 9. □

7. Local and global area variance

Given an interval $I \subset (0, 2\pi)$ we define the local variance

$$\operatorname{Var}(R, I) := \frac{1}{|\Lambda(R, I)|} \sum_{\lambda \in \Lambda(R,I)} \big(A(\lambda) - \operatorname{Mean}(R, I)\big)^2. \tag{9}$$

As in Section 2.2, after subdividing into octants and using the symmetries of $\mathbb{Z}^2$, we may restrict attention to intervals contained in the seventh octant.

Lemma 21. *Let $I \subset (3\pi/2, 7\pi/4)$ and assume that*

$$|I| > \frac{(\log R)^8}{\sqrt{R}}$$

and put $\hat{I} := I - \frac{3\pi}{2} \subset (0, \pi/4)$. Then for every bounded continuous function

$$F : (0, \pi/4) \times \partial_L(S) \to \mathbb{R}$$

we have

$$\frac{1}{|\Lambda(R, I)|} \sum_{\lambda \in \Lambda(R,I)} F(\theta(\lambda), v(\lambda)) = \frac{\displaystyle\int_{\hat{I} \times \partial_L(S)} F(\theta, v)\, d\mu(\theta, v)}{\displaystyle\int_{\hat{I} \times \partial_L(S)} d\mu(\theta, v)} + o(1),$$

where

$$d\mu(\theta, v) = \begin{cases} \sin\theta\, d\theta\, dx, & v = (x, 0), \\ \cos\theta\, d\theta\, dy, & v = (0, y). \end{cases}$$

Proof. We give a sketch based on the proof of Proposition 17. We partition $\hat{I} \times \partial_L(S)$ into admissible shrinking boxes $Q = I \times J$ (i.e., so we may apply Proposition 19), and replace F on Q by its value at a representative point. The counting measure on lattice points pushes forward to μ, uniformly over the partition, and the resulting Riemann sums converge to the corresponding μ-integral. Applying the same argument to $F \equiv 1$ gives the denominator. □

Lemma 22. *For $0 < \theta < \pi/4$, define*

$$g(\theta) := \frac{\tan^3\theta - 5\tan^2\theta + 15\tan\theta + 5}{60(1 + \tan\theta)}.$$

Then

$$\int_{\partial_L(S)} \Big(|B_{\theta,v} \cap S| - \frac{1}{2}\Big)^2 d\mu_\theta(v) = (\sin\theta + \cos\theta)\, g(\theta).$$

Proof. We need an explicit expression for the area in the fundamental domain above a specific line. Fix $0<\theta<\pi/4$, and write $m=\tan\theta$. A line of slope m is written $y=mx+b$. The pushforward of $d\mu_\theta$ to the y-intercept parameter b is

$$d\mu_\theta \mapsto \cos\theta\, db \qquad \text{on } [-m,1].$$

Let $A(b,\theta)$ denote the area of the part of $S=[0,1]^2$ lying above the line $y=mx+b$. Then

$$A(b,\theta)=\begin{cases} 1-\dfrac{(m+b)^2}{2m}, & -m\le b\le 0,\\ 1-b-\dfrac{m}{2}, & 0\le b\le 1-m,\\ \dfrac{(1-b)^2}{2m}, & 1-m\le b\le 1.\end{cases} \tag{10}$$

Therefore

$$\int_{\partial_L(S)}\Big(|B_{\theta,v}\cap S|-\frac{1}{2}\Big)^2\,d\mu_\theta(v)=\cos\theta\int_{-m}^{1}\Big(A(b,\theta)-\frac{1}{2}\Big)^2\,db.$$

A direct computation on the three intervals gives

$$\int_{-m}^{1}\Big(A(b,\theta)-\frac{1}{2}\Big)^2\,db=\frac{m^3-5m^2+15m+5}{60}.$$

□

Proposition 23 (Local Variance)**.** *Let*

$$|I|>\frac{(\log R)^8}{\sqrt{R}}.$$

Assume that $I\subset(3\pi/2,7\pi/4)$ *and write* $\hat{I}:=I-\frac{3\pi}{2}\subset(0,\pi/4)$*. Then*

$$\mathrm{Var}(R,I)=\frac{\displaystyle\int_{\hat{I}}g(\theta)(\sin\theta+\cos\theta)\,d\theta}{\displaystyle\int_{\hat{I}}(\sin\theta+\cos\theta)\,d\theta}+o(1).$$

If in addition $|\hat{I}|\to 0$ *and* $\theta_0\in\hat{I}$*, then*

$$\mathrm{Var}(R,I)\to g(\theta_0).$$

Proof. By (9) and the local mean result (cf. Proposition 17), we have

$$\mathrm{Var}(R,I)=\frac{1}{|\Lambda(R,I)|}\sum_{\lambda\in\Lambda(R,I)}\Big(A(\lambda)-\frac{1}{2}\Big)^2+o(1).$$

Lemma 21 then yields

$$\mathrm{Var}(R, I) = \frac{\displaystyle\int_{J_R \times \partial_L(S)} \left(|B_{\theta,v} \cap S| - \frac{1}{2}\right)^2 d\mu(\theta, v)}{\displaystyle\int_{J_R \times \partial_L(S)} d\mu(\theta, v)} + o(1).$$

Now integrate first in v. By Lemma 22,

$$\int_{\partial_L(S)} \left(|B_{\theta,v} \cap S| - \frac{1}{2}\right)^2 d\mu_\theta(v) = (\sin\theta + \cos\theta)\, g(\theta).$$

Therefore

$$\int_{J_R \times \partial_L(S)} \left(|B_{\theta,v} \cap S| - \frac{1}{2}\right)^2 d\mu(\theta, v) = \int_{J_R} g(\theta)(\sin\theta + \cos\theta)\, d\theta,$$

while

$$\int_{J_R \times \partial_L(S)} 1\, d\mu(\theta, v) = \int_{J_R} (\sin\theta + \cos\theta)\, d\theta.$$

Substituting these identities proves the first claim. The second follows by continuity of g. □

Corollary 24 (Global Variance)**.** *We have*

$$\mathrm{Var}(R, (3\pi/2, 7\pi/4)) \to c_{\mathrm{glob}},$$

where

$$c_{\mathrm{glob}} = \frac{13}{60} - \frac{1}{12}\log(1 + \sqrt{2}) - \frac{1}{30\sqrt{2}}.$$

Proof. Applying Proposition 23 and implementing the substitution $u = \tan\theta$, gives

$$c_{\mathrm{glob}} = \frac{1}{60}\int_0^1 \frac{u^3 - 5u^2 + 15u + 5}{(1 + u^2)^{3/2}}\, du.$$

A direct computation gives the desired result. □

8. Limiting distribution

Proof of Proposition 3 . Following the same lines of the proof of Proposition 23 the result is reduced to the following computation. Fix $0 < \theta < \pi/4$, and write $m = \tan\theta$. Let $A = A(b, \theta)$ denote the area of the part of $S = [0, 1]^2$ lying above the line $y - mx = b$. Then, by (10), it follows that the conditional density of A for fixed m is

$$f_m(a) = \begin{cases} \dfrac{\sqrt{m}}{(1+m)\sqrt{2a}}, & 0 < a < \dfrac{m}{2}, \\ \dfrac{1}{1+m}, & \dfrac{m}{2} < a < 1 - \dfrac{m}{2}, \\ \dfrac{\sqrt{m}}{(1+m)\sqrt{2(1-a)}}, & 1 - \dfrac{m}{2} < a < 1, \end{cases}$$

and in particular $f_m(a) = f_m(1-a)$.

We now average over θ. Since

$$\int_0^{\pi/4} (\sin\theta + \cos\theta)\, d\theta = 1,$$

the angle variable is distributed with density $(\sin\theta + \cos\theta)\, d\theta$. Passing to $m = \tan\theta$, we have

$$d\theta = \frac{dm}{1+m^2}, \qquad \sin\theta + \cos\theta = \frac{1+m}{\sqrt{1+m^2}},$$

so the measure becomes

$$\frac{1+m}{(1+m^2)^{3/2}}\, dm, \qquad 0 < m < 1.$$

Hence

$$f(a) = \int_0^1 f_m(a)\, \frac{1+m}{(1+m^2)^{3/2}}\, dm.$$

Substituting the formula for $f_m(a)$ and simplifying gives, for $0 < a < \frac{1}{2}$,

$$f(a) = \int_0^{2a} \frac{dm}{(1+m^2)^{3/2}} + \int_{2a}^1 \frac{\sqrt{m}}{\sqrt{2a}\,(1+m^2)^{3/2}}\, dm.$$

Since each f_m is symmetric, so is f, and therefore $f(a) = f(1-a)$ for all $0 < a < 1$. This proves the formula for the density. $\square$

9. Further questions

The results of this paper suggest several possible extensions. First, one may replace the square fundamental domain $S = [0,1)^2$ by a different fundamental domain for $\mathbb{Z}^2$, or more generally for an arbitrary planar lattice. The case of polygonal fundamental domains appears particularly interesting: the area functional is still piecewise analytic, but the sequence of sides through which the curve enters may have a more complicated symbolic structure. In particular, the order-stability property does not always hold.

A second direction is to replace the disk by other convex domains. For strictly convex domains whose boundary has curvature uniformly bounded way from zero, we would expect similar qualitative behavior.

A basic problem is to characterize domains (and fundamental domains) which have asymptotically vanishing area correlations.

Appendix A. A historical overview

We give a short historical overview of Gauss original formulation for the circle problem. In his study of binary quadratic forms [1, I., art. 4], he introduced what is now known as "the Gauss circle problem", and states that if the area of a figure scales as k^2 and the length of the perimeter scales as k then the number of the lattice points inside asymptotically equals the area. He gives the circle as an example of a figure, with the asymptotic $|B(R) \cap \mathbb{Z}^2| \sim \pi R^2$, but provides no error term. Furthermore the proof is (roughly) "suppressed, hastening to more difficult matters"

> Ceterum si operae pretium esset, facile demonstrationem illius theorematis antiquo rigore absolvere possemus, quam tamen hocce quidem loco supprimere maluimus ad difficiliora properantes.

In [1, II., art. 2] Gauss states that a figure delimited by a closed curve can be divided into squares inside the figure and squares intersecting the perimeter. He then states that the area of the figure can be approximated up to any degree of precision by the area of the internal squares, taking small enough squares (with some implicit assumptions on the uniformity of the boundary of the figure). To prove this he shows that the number of squares of side a intersected by a (closed) curve or length l is at most $4\left(1+\frac{l}{a}\right)$. This shows that the area of the intersected squares is negligible as $a \to 0$. In [1, II., art. 3] he then uses this result to prove that the number of lattice points (associated to squares of side a), situated inside the figure, multiplied by the area a^2 of the square, is asymptotic to the area of the figure if a tends to zero.

In fact, Gauss proves a result valid for general figures, though without stating precise assumptions on the figure. This, though not explicitly highlighted, gives

$$|B(R) \cap \mathbb{Z}^2| = \pi R^2 + O(R)$$

with an explicit constant.

We remark that many modern sources attribute a slightly different proof to Gauss, namely bounding the lattice points from above and below in terms of the areas of two circles, one of radius $R - 1/\sqrt{2}$, and one of radius $R + 1/\sqrt{2}$. We have not been able to locate such a proof in Gauss's writings.

Appendix B. Heuristics for Hardy's conjecture

Let $N(R) := |B(R) \cap \mathbb{Z}^2|$ and define $E(R) := N(R) - \pi R^2$. In [2], Hardy asserted that "...and it is not unlikely that $E(R) = O(R^{1/2+\epsilon})$ for all positive values of ϵ, though this has never been proved.", and then showed that it is "true on average" in that

$$\frac{1}{x}\int_0^x |E(R)| dR = O(x^{1/2+\epsilon})$$

as $x \to \infty$.

A folklore heuristic is that Hardy's conjecture would follow from square root cancellation when summing over "boundary terms", provided these terms are distributed as some sort of i.i.d. random variables.

Given our setup we can give a probabilistic/dynamical heuristic that exhibits an interesting "perfect cancellation" phenomena between the curved and horizontal part of the boundary of a *quarter circle.* The number of lattice points in the first (half open) quarter circle, i.e., $|\{v \in \mathbb{Z}^2 : |v| \leq R, \arg(v) \in [0, \pi/2)\}|$ then equals $N(R)/4$. With $M(R)$ denoting the number of "internal" lattice points, i.e.,

$$M(R) := |\{(x, y) \in \mathbb{Z}^2 : x, y > 0, \arg((x, y)) \in [0, \pi/2)\}|$$

we have $N(R)/4 = M(R) + R + O(1)$. Moreover, computing the area of the quarter circle we have

$$M(R) \cdot 1 + \sum_{\lambda \in \Lambda(R,(0,\pi/2))} A(\lambda) = \pi R^2/4$$

which gives

$$N(R)/4 = \pi R^2/4 - R + \sum_{\lambda \in \Lambda(R,(0,\pi/2))} A(\lambda).$$

Since $|\Lambda(R, (0, \pi/2))| = 2R + O(1)$, we obtain

$$N(R) = \pi R^2 + 4 \sum_{\lambda \in \Lambda(R,(0,\pi/2))} \left(A(\lambda) - \frac{1}{2}\right) + O(1).$$

Thus, if the mean area is $1/2 + o(1)$, i.e.,

$$\frac{1}{\Lambda(R, (0, \pi/2))} \sum_{\lambda \in \Lambda(R,(0,\pi/2))} A(\lambda) = 1/2 + o(1)$$

we find that

$$N(R) = \pi R^2 + o(R)$$

which gives a slight improvement on Gauss' error term $O(R)$.

We remark that $|\Lambda(R,(0,\pi/2)| \sim c \cdot R$ for $c \neq 2$ would imply that Gauss' error term $O(R)$ is *sharp*. In fact, any improvement beyond Gauss crucially depends on a leading order cancellation between two boundary terms: one from lattice points on the horizontal axis, the other related to the areas of boundary squares intersected with $B(R)$.

Finally, if $\{A(\lambda) - 1/2\}_{\lambda \in \Lambda(R,(0,\pi/2)}$ "behaves as a collection of I.I.D random variables with mean zero", one would obtain $N(R) = \pi R^2 + O(R^{1/2+\epsilon})$ for any $\epsilon > 0$.

B.1. **Cancellation via trapezoidal integration.** A similar cancellation occurs when counting lattice points by taking the integer part of $\sqrt{R^2 - x^2}$; writing $\rho(x) = 1/2 - \{x\}$ where $\{x\} = x - [x]$ denotes the fractional part of x, one finds a similar phenomena. Namely, following Jameson [4], let $f(x) = \sqrt{R^2 - x^2} - x$ and put $K = [R/\sqrt{2}]$; then $M(R)$, the number of integers in the half open quarter circle equals,

$$2\sum_{k=1}^{K}[f(k)] + K + [R] = 2\sum_{k=1}^{K} f(k) + 2\sum_{k=1}^{K} \rho(f(k)) + [R]$$

and using the trapezoidal rule we have

$$\sum_{k=1}^{K} f(k) = \frac{\pi R^2}{8} - R/2 + O(1)$$

and obtain

$$M(R) = \pi R^2/4 + 2\sum_{k=1}^{K} \rho(f(k)).$$

Again there is essentially perfect cancellation between lattice points on the y-coordinate axis (the term $[R]$ above) and the difference between the $\sum_{k=1}^{K} f(k)$ and the integral $\int_0^K f(t)\,dt$.

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