EVIDENCE FOR THE DYNAMICAL BRAUER-MANIN CRITERION

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ABSTRACT. Let $\varphi\colon X \to X$ be a morphism of a variety over a number field K. We consider local conditions and a "Brauer-Manin" condition, defined by Hsia and Silverman, for the orbit of a point $P \in X(K)$ to be disjoint from a subvariety $V \subseteq X$, i.e., for $V \cap \mathcal{O}_\varphi(P) = \emptyset$. We provide evidence that the dynamical Brauer-Manin condition is sufficient to explain the lack of points in the intersection $V \cap \mathcal{O}_\varphi(P)$; this evidence stems from a probabilistic argument as well as unconditional results in the case of étale maps.

1. Introduction

In recent work, Hsia and Silverman (Hsia and Silverman, 2009) ask a dynamical question in analogy with a question of Scharaschkin (Scharaschkin, 1999); the dynamical question is as follows. Let $\varphi\colon X \to X$ be a self-morphism of a variety over a number field K, let $V \subseteq X$ be a subvariety, and let $P \in X(K)$. The orbit of P under ϕ is defined by

$$\mathcal{O}_\varphi(P) := \{P, \varphi(P), \varphi(\varphi(P)), \ldots\}.$$

We also let $\mathbb{A}_K$ be the ring of adeles of K and $\mathcal{C}(-)$ denotes the closure in the adelic topology of subsets of $V(\mathbb{A}_K)$. Hsia and Silverman ask whether the closure of the intersection of the orbit $\mathcal{O}_\varphi(P)$ with the subvariety V is equal to the intersection of $V(\mathbb{A}_K)$ with the closure of $\mathcal{O}_\varphi(P)$ in the adelic topology, i.e. whether

$$\mathcal{C}(V(K) \cap \mathcal{O}_\varphi(P)) = V(\mathbb{A}_K) \cap \mathcal{C}(\mathcal{O}_\varphi(P)).$$

The purpose of this paper is to give some justification to the assertion that a closely related question has a positive answer. In particular, when $K = \mathbb{Q}$, (and a choice of an integral model for V is made) we give evidence for the assertion that for fixed P, V

$$V(K) \cap \mathcal{O}_\varphi(P) = \emptyset \Rightarrow \exists m \in \mathbb{Z}, m > 1, V(\mathbb{Z}/m\mathbb{Z}) \cap (\mathcal{O}_\varphi(P) \bmod m) = \emptyset \tag{1.1}$$

holds for "sufficiently generic" φ

Our evidence is twofold: an analogous result in a probabilistic model, and unconditional results in the case that φ is étale and V is φ^k-invariant or φ-preperiodic.

The probabilistic model — inspired by similar work of Poonen (Poonen, 2006) for the original question of Scharaschkin — is developed in Section 2, and it suggests the following:

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(i) For all sufficiently large prime p, it is highly probable that, provided $\dim V > \dim X/2$,

$$V(\mathbb{Z}/p\mathbb{Z}) \cap (\mathcal{O}_\varphi(P) \bmod p) \neq \emptyset$$

(cf. Proposition 2.4.) It thus seems unlikely that one would be able to show that $V(K) \cap \mathcal{O}_\varphi(P) = \emptyset$ by exhibiting a prime p so that the modulo p intersection is empty, even when it is expected that the modulo m intersection is empty for some m (as discussed below). In other words, a "Chinese remainder theorem" does not work for orbits.

(ii) On the other hand, if $V(K) \cap \mathcal{O}_\varphi(P) = \emptyset$, then there very likely exist infinitely many squarefree integers m such that

$$V(\mathbb{Z}/m\mathbb{Z}) \cap C_m = \emptyset,$$

where C_m denotes the cyclic part of $\mathcal{O}_\varphi(P)$ in $X(\mathbb{Z}/m\mathbb{Z})$ (cf. Proposition 2.6.)

Remark 1.1. If the intersection of the orbit $\mathcal{O}_\varphi(P)$ modulo m with $V(\mathbb{Z}/m\mathbb{Z})$ is contained only in the tail of $\mathcal{O}_\varphi(P)$ mod m, then there exists N_0 such that for all $n > N_0$, $\varphi^n(P) \bmod m \notin V(\mathbb{Z}/m\mathbb{Z})$ (here φ^n denotes the composition of φ with itself n times). Therefore $V(K) \cap \mathcal{O}_\varphi(P)$ is contained in

$$\{P, \varphi(P), \varphi^2(P), \ldots, \varphi^{N_0}(P)\},$$

and so can be determined by a finite computation.

In contrast, if the cyclic part of $\mathcal{O}_\varphi(P)$ modulo m intersects $V(\mathbb{Z}/m\mathbb{Z})$, then there are infinitely many integers n such that $\varphi^n(P) \bmod m \in V(\mathbb{Z}/m\mathbb{Z})$. Hence, we cannot *a priori* show that $V(K) \cap \mathcal{O}_\varphi(P)$ is contained in a finite set. Therefore, it is reasonable to only consider the intersection with the cyclic part of the orbit when trying to formulate and give heuristic evidence to a local criterion for the intersection of V with the orbit of P to be empty.

In Section 3, we provide numerical evidence for the randomness assumptions needed in the heuristic argument from Section 2. We also describe experiments on randomly generated morphisms of $\mathbb{A}^5$ which support the argument that (1.1) holds for many φ.

The unconditional results are the focus of Section 4. Assume that X is quasi-projective, that φ is étale, and that $\varphi^k(V) \subseteq V$, i.e., that V is φ^k-*invariant*, for some positive integer k. Under these assumptions we show that if $V \cap \mathcal{O}_\varphi(P) = \emptyset$, then for all but finitely many primes p, there exists an $n = n(p)$ such that $V(\mathbb{Z}/p^n\mathbb{Z}) \cap \mathcal{O}_\varphi(P) \bmod p^n = \emptyset$.

An irreducible subvariety W of X is called *preperiodic* (or φ-preperiodic if the morphism is not clear from the context) if $\varphi^{k_0+k}(W) = \varphi^{k_0}(W)$ for some integers $k_0 \geq 0$ and $k > 0$. If every irreducible component of V is preperiodic, X is quasi-projective, and φ is étale and closed, then we obtain the same result. In other words, we prove

Theorem 1.2. *Let X be a quasi-projective variety over a global field K and V a closed subvariety of X. Assume that φ is étale, and either (1) that V is φ^k-invariant, for some $k \geq 1$ or (2) that φ is closed and every irreducible component of V is φ-preperiodic. If $P \in X(K)$ is such that $V(K) \cap \mathcal{O}_\varphi(P) = \emptyset$ then, for all but finitely many primes v,*

$$V(K_v) \cap \mathcal{C}_v(\mathcal{O}_\varphi(P)) = \emptyset,$$

where $\mathcal{C}_v(-)$ denotes closure in the v-adic topology.

In Section 4.6, we discuss whether the assumption that V is preperiodic or φ^k-invariant can be weakened in any way.

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2. Probabilistic Proof

Let X be a projective variety over $\mathbb{Q}$, let $V \subset X$ be a geometrically irreducible[1] closed $\mathbb{Q}$-subvariety, and let $\varphi\colon X \to X$ be a $\mathbb{Q}$-morphism. We write

$$D := \dim(X), \quad \text{ and } \quad d := \dim(V).$$

The goal of this section is to give probabilistic evidence for the assertion that if $V \cap \mathcal{O}_\varphi(P) = \emptyset$ and φ is "sufficiently generic", then there exists positive squarefree integers m such that $V(\mathbb{Z}/m\mathbb{Z})$ is disjoint from the cyclic part of $\mathcal{O}_\varphi(P)$ mod m. Specifically, we model the cyclic part of $\mathcal{O}_\varphi(P)$ mod m by certain random subsets C'_m of $X(\mathbb{Z}/m\mathbb{Z})$, and then prove that the probability that $V(\mathbb{Z}/m\mathbb{Z}) \cap C'_m = \emptyset$ approaches 1 as $m \to \infty$ along a certain subsequence. To make this more precise, we fix some further notation.

Fix a finite set of primes S such that X and V extend to flat projective models $\mathscr{X}$ and $\mathscr{V}$ over $\mathbb{Z}_S$, the ring of S-integers, i.e. rational numbers whose denominators are divisible only by primes in S. After possibly enlarging S, we also assume that φ extends to a morphism $\tilde{\varphi}\colon \mathscr{X} \to \mathscr{X}$ and that $P : \operatorname{Spec} K \to X$ extends to $\mathscr{P} : \operatorname{Spec} \mathbb{Z}_S \to \mathscr{X}$. Then for any integer m which is relatively prime to all elements of S, we may consider the base change $X_m := \mathscr{X} \times_{\mathbb{Z}_S} \mathbb{Z}/m\mathbb{Z}, V_m := \mathscr{V} \times_{\mathbb{Z}_S} \mathbb{Z}/m\mathbb{Z}$ as well as the restrictions $\varphi_m := \tilde{\varphi}|_{X_m}\colon X_m \to X_m, P_m := \mathscr{P}|_{\operatorname{Spec}\mathbb{Z}/m\mathbb{Z}} : \operatorname{Spec}\mathbb{Z}/m\mathbb{Z} \to X_m$. We will often abuse notation and write $V(\mathbb{Z}/m\mathbb{Z})$ for $V_m(\mathbb{Z}/m\mathbb{Z})$.

We write

$$O_m := \{P_m, \varphi_m(P_m), \varphi_m(\varphi_m(P_m)), \ldots\}.$$

As O_m is contained in the finite set $X_m(\mathbb{Z}/m\mathbb{Z})$, there is some pair of non-negative integers $k_0 < k_1$ such that $\varphi_m^{k_0}(P_m) = \varphi_m^{k_1}(P_m)$. Let k_0 be the minimal such integer; then we define the cyclic part of O_m as

$$C_m := \{\varphi_m^{k_0}(P_m), \varphi_m^{k_0+1}(P_m), \varphi_m^{k_0+2}(P_m), \ldots\}.$$

Our probabilistic model is motivated by the following heuristics for C_m and O_m.

(i) The reduction of morphisms $\varphi\colon X \to X$ modulo p behave like random maps on a finite set $X(\mathbb{F}_p)$.

[1]Since $\mathcal{O}_\varphi(P) \subset X(\mathbb{Q})$, the intersection $V(\mathbb{Q}) \cap \mathcal{O}_\varphi(P)$ is contained in a finite union of geometrically irreducible closed $\mathbb{Q}$-subvarieties $V_i \subset V$. Therefore there is no loss of generality in restricting to geometrically irreducible subvarieties V.

(ii) For any $y \in X(\mathbb{F}_p)$, the condition that $y \in O_p$ is independent from the condition that $y \in V(\mathbb{F}_p)$.

(iii) For any $y \in X(\mathbb{Q}) \setminus V(\mathbb{Q})$, the condition that $y \pmod p \in V(\mathbb{F}_p)$ is independent from the condition that $y \pmod q \in V(\mathbb{F}_q)$ for primes $p \neq q$. In particular, if $V \cap \mathcal{O}_\varphi(P) = \emptyset$, then the condition that $\varphi^n(P) \pmod p \in V(\mathbb{F}_p)$ is independent from the condition that $\varphi^n(P) \pmod q \in V(\mathbb{F}_p)$.

2.1. **The probabilistic model.** For each prime $p \notin S$, let $C'_p \subset O'_p \subset X(\mathbb{F}_p)$ be randomly selected subsets of $X(\mathbb{F}_p)$, subject to the cardinality conditions $|C'_p| = |C_p|$ and $|O'_p| = |O_p|$, together with random identification $\mathbb{Z}/|C'_p|\mathbb{Z} \xrightarrow{\sim} C'_p$. For any squarefree integer m which is coprime to all elements of S, we may consider the composition

$$\mathbb{Z} \to \prod_{p|m} \mathbb{Z}/|C'_p|\mathbb{Z} \xrightarrow{\sim} \prod_{p|m} C'_p \subset \prod_{p|m} X(\mathbb{F}_p) \to X(\mathbb{Z}/m\mathbb{Z});$$

note that this factors through $\mathbb{Z}/(\mathrm{lcm}_{p|m}|C'_p|\mathbb{Z})$. Thus, we may define $C'_m \subset X(\mathbb{Z}/m\mathbb{Z})$ as the image of $\mathbb{Z}/(\mathrm{lcm}_{p|m}|C'_p|\mathbb{Z})$ in $X(\mathbb{Z}/m\mathbb{Z})$ under the above map. Notice that this definition implies that $|C_m| = |C'_m|$.

As the notation suggests, we would like to think of C'_m as modeling the behavior of C_m in the case that $V \cap \mathcal{O}_\varphi(P) = \emptyset$.

Lemma 2.1. *Assume that as $p \to \infty$,*

$$|O'_p| = |O_p| = p^{D/2+o(1)}, \quad \text{and} \quad |C'_p| = |C_p| = p^{D/2+o(1)}. \tag{2.1}$$

Then, we have the following properties.

(i) *Independence between O'_p and $V(\mathbb{F}_p)$: As $p \to \infty$,*

$$\begin{aligned} \mathrm{Prob}(V(\mathbb{F}_p) \cap O'_p = \emptyset) &= (1 - 1/p^{D-d+o(1)})^{|O'_p|}, \text{ and} \\ \mathrm{Prob}(V(\mathbb{F}_p) \cap C'_p = \emptyset) &= (1 - 1/p^{D-d+o(1)})^{|C'_p|}. \end{aligned} \tag{2.2}$$

(ii) *Asymptotic independence modulo large primes: as $T \to \infty$,*

$$\begin{aligned} \mathrm{Prob}(V(\mathbb{F}_p) \cap O'_p \neq \emptyset \ \forall p > T) &= \textstyle\prod_{p>T} \mathrm{Prob}(V(\mathbb{F}_p) \cap O'_p \neq \emptyset), \text{ and} \\ \mathrm{Prob}(V(\mathbb{F}_p) \cap C'_p \neq \emptyset \ \forall p > T) &= \textstyle\prod_{p>T} \mathrm{Prob}(V(\mathbb{F}_p) \cap C'_p \neq \emptyset). \end{aligned} \tag{2.3}$$

(iii) *Independence modulo squarefree m: For all $x \in V(\mathbb{Z}/m\mathbb{Z})$,*

$$\mathrm{Prob}(x \in C'_m) = |C'_m|/m^{D+o(1)}, \tag{2.4}$$

as the smallest prime factor of m tends to infinity.

Proof. By the Weil conjectures, $|X(\mathbb{F}_p)| = p^{D+o(1)}$ and $|V(\mathbb{F}_p)| = p^{d+o(1)}$ (as V is geometrically irreducible modulo p for p sufficiently large) and we find that the probability that a randomly selected point in $X(\mathbb{F}_p)$ lies in $V(\mathbb{F}_p)$ equals $1/p^{D-d+o(1)}$. Thus, if $O'_p \subset X(\mathbb{F}_p)$ is a random subset with given cardinality, then

$$\begin{aligned} \mathrm{Prob}(V(\mathbb{F}_p) \cap O'_p = \emptyset) &= \binom{|X(\mathbb{F}_p)| - |V(\mathbb{F}_p)|}{|O'_p|} \Big/ \binom{|X(\mathbb{F}_p)|}{|O'_p|} = \prod_{i=0}^{|O'_p|-1} \frac{|X(\mathbb{F}_p)| - |V(\mathbb{F}_p)| - i}{|X(\mathbb{F}_p)| - i} \\ &= \prod_{i=0}^{|O'_p|-1} \left(1 - \frac{|V(\mathbb{F}_p)|}{|X(\mathbb{F}_p)|} \cdot (1 + O(i/|X(\mathbb{F}_p)|)\right) = (1 - 1/p^{D-d+o(1)})^{|O'_p|}, \end{aligned}$$

since $\frac{|V(\mathbb{F}_p)|}{|X(\mathbb{F}_p)|} \cdot (1 + O(i/|X(\mathbb{F}_p)|) = 1/p^{D-d+o(1)}$ for $i < |O'_p| = o(|X(\mathbb{F}_p)|))$. The same asymptotics hold for $\mathrm{Prob}(V(\mathbb{F}_p) \cap C'_p = \emptyset)$.

Property (ii) holds since the sets O'_p for different p are chosen independently, and the same for C'_p.

We may select independent random subsets $C'_p \subset X(\mathbb{F}_p)$ as follows: for each prime p let σ_p be a random permutation (chosen independently for different p) of $X(\mathbb{F}_p)$, and let $C'_p = \sigma_p(C_p)$. Hence $\mathrm{Prob}(x \in C'_m)$ is the same as the probability of a randomly selected element of $X(\mathbb{Z}/m\mathbb{Z})$ lying in C_m; in turn, this probability equals $|C_m|/m^{D+o(1)} = |C'_m|/m^{D+o(1)}$. This proves (iii). □

Remark 2.2. If A is a (large) finite set, and $f : A \to A$ is a map chosen uniformly at random from the set of all possibilities, then the cardinality of the forward orbit of a random starting point is likely to be of size $|A|^{1/2+o(1)}$ (Flajolet and Odlyzko, 1990), as $|A| \to \infty$. This motivates assumption (2.1).

Remark 2.3. We warn the reader that there are maps of special type for which the random map heuristic does not apply. For example, for any linear automorphism φ of $\mathbb{P}^n$ which is represented by a semisimple matrix, there is a positive density set of primes for which $O_p|(p-1)$. Specifically, one can take the set of primes for which the characteristic polynomial of $A_\varphi \in \mathrm{PGL}_{n+1}$ splits completely.

2.2. **Nonempty intersections modulo** p. Consider the case where V has sufficiently small codimension, say 1. Then we may expect that there is a $1/p$ chance of $\varphi^n(P)$ landing in V modulo p for any n. Furthermore, if the orbit is of length $p^{D/2}$, as we would expect, then the likelihood of $V(\mathbb{F}_p) \cap O'_p = \emptyset$ should be given by $(1 - 1/p)^{|O'_p|} = \exp(-(1 + o(1))|O'_p|/p)$, assuming p is sufficiently large. Thus, if the orbits are long, then one expects "accidental" intersections modulo p, even if $V \cap \mathcal{O}_\varphi(P) = \emptyset$.

With our probabilistic model, we are able to make this precise.

Proposition 2.4. *Assume that assumptions* (2.1)–(2.3) *hold, and that* $d > D/2$. *Then, as* $T \to \infty$,

$$\mathrm{Prob}(V(\mathbb{F}_p) \cap O'_p \neq \emptyset \ \forall p > T) = 1 - e^{-T^{d-D/2+o(1)}} = 1 - o(1).$$

Proof. By assumption (2.1), $|O'_p| = p^{D/2+o(1)}$, hence assumption (2.2) gives that

$$\begin{aligned}\mathrm{Prob}(V(\mathbb{F}_p) \cap O'_p = \emptyset) &= \left(1 - p^{d-D+o(1)}\right)^{|O'_p|} = \left(1 - p^{d-D+o(1)}\right)^{p^{D/2+o(1)}} \\ &= e^{-p^{D/2-(D-d)+o(1)}} = e^{-p^{d-D/2+o(1)}}.\end{aligned}$$

In particular,

$$\mathrm{Prob}(V(\mathbb{F}_p) \cap O'_p \neq \emptyset) = 1 - e^{-p^{d-D/2+o(1)}},$$

and therefore, by assumption (2.3), as $T \to \infty$,

$$\mathrm{Prob}(V(\mathbb{F}_p) \cap O'_p \neq \emptyset \ \forall p > T) = \prod_{p>T}(1 - e^{-p^{d-D/2+o(1)}}) = 1 - e^{-T^{d-D/2+o(1)}} = 1 - o(1).$$

□

2.3. **Empty intersections for some composite m.** The situation is quite different over composite integers m. Indeed, the main result of this section is that there exist squarefree integers m such that the probability that $V(\mathbb{Z}/m\mathbb{Z})$ and C'_m are disjoint is arbitrarily close to 1.

We begin by recalling some background on smooth numbers.

Definition 2.5. *An integer n is y-smooth if all primes p dividing n are bounded above by y. Define*

$$\psi(x,y) := |\{n \le x : n \text{ is } y-\text{smooth}\}|.$$

Smooth integers have the following well-known distribution (Tenenbaum and Mendès France, 2000, Thm. 10, p. 97) for $\alpha \in (0,1)$ and x tending to infinity,

$$\psi(x, x^\alpha) = (1+o(1)) \cdot x \cdot \rho(u)$$

where $u := \log x / \log x^\alpha = 1/\alpha$, and $\rho(u) \in (0,1)$ for $u \in (1,\infty)$.

Our analysis will be based on the following heuristic: that $|C'_p|$ has the same "likelihood" of being smooth as a random integer of the same size. By (2.1), $|C'_p| = p^{D/2+o(1)}$, so the heuristic implies that the density of primes p for which $|C'_p| = p^{D/2+o(1)}$ is p^α-smooth equals $\rho(u)$ where $u = \log p^{D/2+o(1)} / \log p^\alpha = D(1+o(1))/(2\alpha)$. In particular, we expect that as $x \to \infty$,

$$|\{p \in [\log x, x] : |C'_p| \text{ is } x^\alpha\text{-smooth}\}| = (1+o(1)) \cdot \pi(x) \cdot \rho\left(\frac{D(1+o(1))}{2\alpha}\right),$$

which, by partial summation, implies that

$$\prod_{\substack{p\in[\log x, x] \\ |C'_p| \text{ is } x^\alpha\text{-smooth}}} p = \exp\left(\sum_{\substack{p\in[\log x, x] \\ |C'_p| \text{ is } x^\alpha\text{-smooth}}} \log p\right) = \exp\left(x \cdot \rho\left(\frac{D(1+o(1))}{2\alpha}\right)\right).$$

This heuristic leads us to the following precise cycle length smoothness assumption: For α and D fixed and $x \to \infty$,

$$\prod_{\substack{p\in[\log x, x] \\ |C'_p| \text{ is } x^\alpha\text{-smooth}}} p = \exp\left(x \cdot \rho\left(\frac{D(1+o(1))}{2\alpha}\right)\right). \tag{2.5}$$

Proposition 2.6. *Assume that (2.1) and (2.5) hold, and that $d < D$. Then there exists a sequence of* squarefree *integers m such that*

$$\mathrm{Prob}(V(\mathbb{Z}/m\mathbb{Z}) \cap C'_m = \emptyset) = 1 - o(1)$$

as $m \to \infty$.

Proof. Define

$$m_{x,\alpha} := \prod_{\substack{p\in[\log x, x] \\ |C'_p| \text{ is } x^\alpha\text{-smooth}}} p.$$

Since for any squarefree integer M, $|C'_M| = \mathrm{lcm}_{p|M}\, |C'_p|$, we have, as $x \to \infty$,

$$|C'_{m_{x,\alpha}}| \le \prod_{p \le x^\alpha} p^{\log x^{D/2+o(1)}/\log(p)} = \exp\left(\sum_{p\le x^\alpha} (D/2 + o(1)) \log x\right)$$
$$= \exp\left((D/2 + o(1)) \cdot \log x \cdot \pi(x^\alpha)\right) = \exp\left(x^\alpha \cdot \frac{D\cdot(1+o(1))}{2\alpha}\right).$$

In particular, for $m = m_{x,1/3}$, we have that $|C'_m| = \exp(O(x^{1/3})) = m^{o(1)}$.

To bound $\mathrm{Prob}(V(\mathbb{Z}/m\mathbb{Z}) \cap C'_m \ne \emptyset)$ we will use Markov's inequality. For each $x \in V(\mathbb{Z}/m\mathbb{Z})$ define a random variable B_x by letting $B_x = 1$ if $x \in C'_m$, otherwise let $B_x = 0$. By (2.4),

$$\mathbb{E}(B_x) = 0 \cdot (1 - |C'_m|/m^{D+o(1)}) + 1 \cdot |C'_m|/m^{D+o(1)} = |C'_m|/m^{D+o(1)}$$

for all $x \in V(\mathbb{Z}/m\mathbb{Z})$, thus

$$\mathbb{E}\left(|V(\mathbb{Z}/m\mathbb{Z}) \cap C'_m|\right) = \mathbb{E}\left(\sum_{x\in V(\mathbb{Z}/m\mathbb{Z})} B_x\right)$$
$$= \sum_{x\in V(\mathbb{Z}/m\mathbb{Z})} \mathbb{E}(B_x) = |V(\mathbb{Z}/m\mathbb{Z})| \cdot |C'_m|/m^{D+o(1)} = m^{d-D+o(1)} = o(1).$$

Markov's inequality then gives

$$\mathrm{Prob}(|V(\mathbb{Z}/m\mathbb{Z}) \cap C'_m| \ge 1) \le \frac{\mathbb{E}\left(|V(\mathbb{Z}/m\mathbb{Z}) \cap C'_m|\right)}{1} = o(1)$$

and thus $\mathrm{Prob}(V(\mathbb{Z}/m\mathbb{Z}) \cap C'_m = \emptyset) = 1 - o(1)$, as $x \to \infty$. □

3. Computations

3.1. **Cycle length smoothness assumption.** We ran experiments, detailed below, to justify the assumption (2.5) on the smoothness of the cycle lengths. Our experiments do not confirm this assumption. Fortunately, it is clear from the proof of Proposition 2.6 that we only need that the cycle lengths be at least as smooth as the prediction (2.5) and this is what we see in the experiments. We also found some maps with special properties for which the cycle lengths are even smoother. We conjecture that cycle lengths are at least as smooth as the prediction (2.5) in all cases but we don't know how to explain the extra smoothness shown in the experiments.

We considered three rational maps:

$$\varphi\colon \mathbb{P}^1 \to \mathbb{P}^1, \quad (x:y) \mapsto (x^2+5y^2 : y^2),$$
$$\psi\colon \mathbb{P}^2 \to \mathbb{P}^2, \quad (x:y:z) \mapsto \left(x^2+y^2 : x^2+3y^2-2xy+z^2 : z^2\right), \text{ and}$$
$$\sigma\colon \mathbb{P}^3 \to \mathbb{P}^3, \quad (x:y:z:w) \mapsto \left(x^2+y^2-z^2+yw+w^2 : x^2-xy+xz+2w^2 :\right.$$
$$\left. z^2-yz+xz+3w^2 : w^2\right).$$

For each prime less than 100000, 500000, and 1000000 respectively we computed C_p, the length of the periodic cycle length of $[1:1]$, $[1:1:1]$, or $[1:1:1:1]$ under φ, ψ, and σ

respectively in $\mathbb{F}_p$. Setting $\alpha = 1/3$, we computed $S(x) := \left[\prod_{\substack{p \leq x \\ |C_p'| \text{ is } x^{\alpha}\text{-smooth}}} p\right]$ at each prime in the range specified. We then created the graphs below which compare $\log S(x)$ to the predicted value of $x \cdot \rho(u)$ where $u = \frac{D}{2\alpha}$ and ρ is the Dickman ρ-function. Recall that assumption (2.5) states that $\log S(x)$ should behave linearly; the graphs (Figures 1, 2, and 3 below) support this assumption. The data appear approximately linear for large enough x, with slope at least as big as predicted. All computations were performed using C and Sage 4.8 (Stein, 2013).

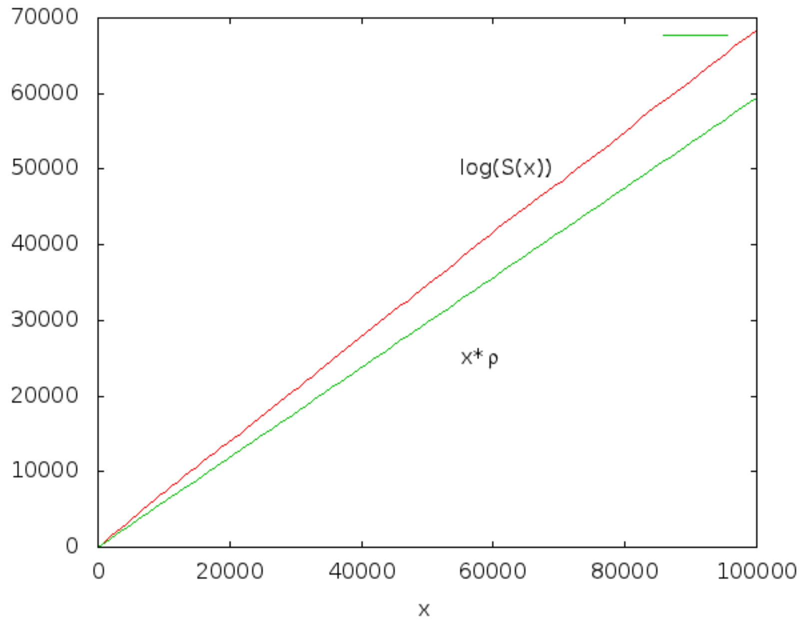

FIGURE 1. $\log S(x)$ for the map φ, and $x < 100000$

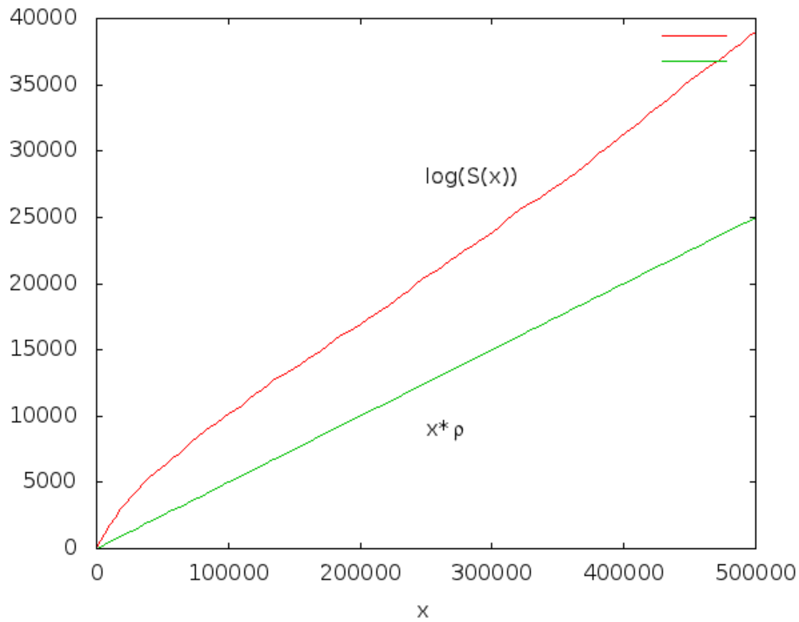

FIGURE 2. $\log S(x)$: for the map ψ, and $x < 500000$

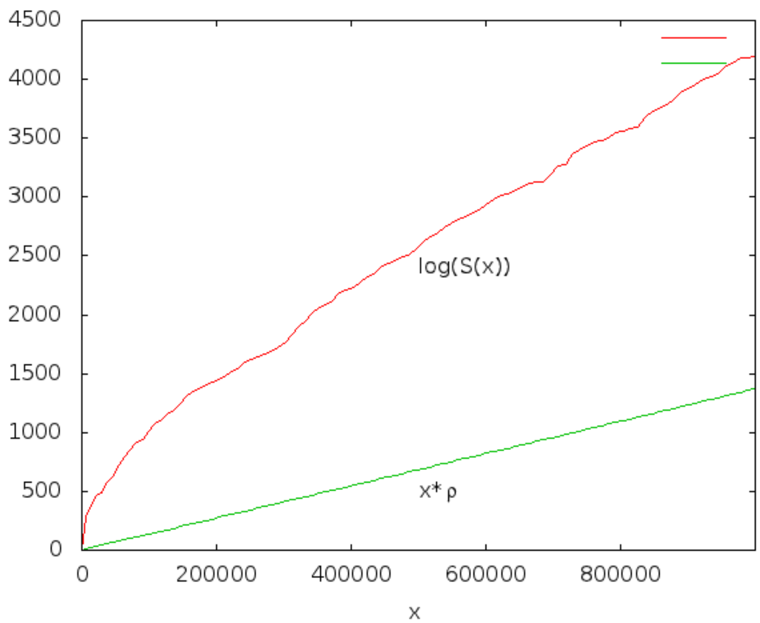

FIGURE 3. $\log S(x)$ for the map σ, and $x < 1000000$

3.2. **Experiments.** In this section, we let $X = \mathbb{A}^5_{(v:w:x:y:z)}$ and let $V = V(1 - v^2 - w^2 - x^2 - y^2 - z^2)$. Fix a point $P \in X(\mathbb{Z})$ and consider a morphism $\phi\colon X \to X$ with integer coefficients; then $\mathcal{O}_\phi(P) \subseteq X(\mathbb{Z})$. As V contains few integral points, namely only those points with exactly one coordinate equal to ± 1 and the remaining coordinates 0, one expects the intersection $V \cap \mathcal{O}_\phi(P)$ to be empty. Thus, by the arguments in Section 2, we expect to find a positive integer m such that $V(\mathbb{Z}/m\mathbb{Z}) \cap O_m(P) = \emptyset$.

We considered a fixed integer starting point $P := (3 : 10 : -4 : 8 : 6)$ and 500 morphisms $X \to X$ given by 5 quadratic polynomials with coefficients chosen randomly from $[-20, 20]$. For each of these morphisms, we computed whether there exists a prime power $q \leq 2000$ such that the intersection $V(\mathbb{Z}/q\mathbb{Z}) \cap O_q(P)$ was empty. If there was no such q, then we computed whether there exists a composite integer $m \leq 2000$ such that $V(\mathbb{Z}/m\mathbb{Z}) \cap O_m(P)$ was empty. The results are as follows:

(i) For 86.8% of these maps, i.e., 434 out of 500, there exists a prime power $q \leq 2000$ such that $V(\mathbb{Z}/q\mathbb{Z}) \cap O_q(P) = \emptyset$.

(ii) For 96.2% of these maps, i.e., 481 out of 500, there exists a positive integer $m \leq 2000$ such that $V(\mathbb{Z}/m\mathbb{Z}) \cap O_m(P) = \emptyset$.

(iii) For 11 of the 19 remaining maps, we concluded that $V \cap \mathcal{O}_\varphi(P) = \emptyset$ by finding an integer $2000 < m < 11500$ such that $V(\mathbb{Z}/m\mathbb{Z}) \cap O_m(P) = \emptyset$. In each of these cases, the smallest such m is supported at more than 1 prime.

(iv) For the remaining 8 maps, $V(\mathbb{Z}/m\mathbb{Z}) \cap O_m(P) \neq \emptyset$ for all $m < 11500$. However, we were still able to conclude that $V \cap \mathcal{O}_\varphi(P) = \emptyset$ by showing that the image of $V(\mathbb{Z})$ modulo 7 is disjoint from $O_7(P)$.

All computations were performed using `Magma` (Bosma et al., 1997).

4. A dynamical Hasse principle for étale morphisms

4.1. **Notation.** Let K be a global field, and let M_K denote its set of places. For a finite set S of places of K containing all the archimedean places, we write $\mathcal{O}_{K,S}$ to denote the ring of S-integers. For all $v \in M_K$, we use K_v to denote the v-adic completion. If v is nonarchimedean, we write $\mathcal{O}_v$, $\mathfrak{m}_v$, and k_v for the valuation ring, maximal ideal, and residue field of v, respectively.

Let X denote a K-variety, i.e., a reduced separated scheme of finite type over K, let $V \subseteq X$ denote a closed K-subvariety, and let $\varphi\colon X \to X$ denote a K-endomorphism. For any K-variety Y, define

$$Y(K, S) := \prod_{v \notin S} Y(K_v). \tag{4.1}$$

We equip $Y(K_v)$ with the v-adic topology and $Y(K, S)$ with the product topology. We view $Y(K)$ as a subset of $Y(K, S)$ via the diagonal embedding. For every subset T of $Y(K, S)$ or $Y(K_v)$, we write $\mathcal{C}(T)$ or $\mathcal{C}_v(T)$ for the closure of T in the product topology or v-adic topology, respectively.

Since Y is separated, $Y(K_v)$, and hence $Y(K, S)$, is Hausdorff. We note that $Y(K, S)$ need not agree with the set of adelic points of Y. For basic terminologies and properties of scheme theory, we refer the readers to (Hartshorne, 1977). For properties of smooth and étale morphisms used throughout this section, we refer the readers to (Grothendieck, 1967).

4.2. **The dynamical Hasse principle for étale maps and preperiodic subvarieties.** For any point $P \in X(K)$, we have the following containments:

$$V(K) \cap \mathcal{O}_\varphi(P) \subseteq V(K, S) \cap \mathcal{C}(\mathcal{O}_\varphi(P)) \subseteq \prod_{v \notin S} V(K_v) \cap \mathcal{C}_v(\mathcal{O}_\varphi(P)). \tag{4.2}$$

Recall the definition by Hsia-Silverman (Hsia and Silverman, 2009, p.237–238) that (X, V, φ) is said to be dynamical Brauer-Manin S-unobstructed if the leftmost containment is an equality for every $P \in X(K)$ satisfying $V^{\mathrm{pp}} \cap \mathcal{O}_\varphi(P) = \emptyset$, where V^{pp} is the union of all positive dimensional preperiodic subvarieties of V. In analogy, we define the dynamical Hasse principle:

Definition 4.1. *The triple (X, V, φ) is said to satisfy the dynamical Hasse principle (over K) if for every $P \in X(K)$ such that $V(K) \cap \mathcal{O}_\varphi(P) = \emptyset$, there exists a place v (depending on P) such that $V(K_v) \cap \mathcal{C}_v(\mathcal{O}_\varphi(P)) = \emptyset$. If there are infinitely many such places v, we say that (X, V, φ) satisfies the strong dynamical Hasse principle.*

When $V = V^{\mathrm{pp}}$, if (X, V, φ) satisfies the strong dynamical Hasse principle then it is immediate that (X, V, S) is Brauer-Manin S-unobstructed for every S. The reason is that for every $P \in X(K)$ such that $V(K) \cap \mathcal{O}_\varphi(P) = \emptyset$, both containments in (4.2) are equalities since all the three sets are the empty set. Our main results in this section are the following:

Theorem 4.2. *Assume that X is quasi-projective, that φ is étale, and that $\varphi^k(V) \subseteq V$ for some $k \in \mathbb{Z}_{>0}$. If $P \in X(K)$ is such that $V(K) \cap \mathcal{O}_\varphi(P) = \emptyset$ then, for all but finitely many primes v,*

$$V(K_v) \cap \mathcal{C}_v(\mathcal{O}_\varphi(P)) = \emptyset.$$

Consequently, (X, V, φ) satisfies the strong dynamical Hasse principle.

We obtain a similar result when every irreducible component of V is φ-preperiodic, under the mild additional assumption that φ is closed:

Theorem 4.3. *Assume that X is quasi-projective and that φ is étale and closed. Let V be a subvariety of X such that every irreducible component is φ-preperiodic. For every $P \in X(K)$, if $V(K) \cap \mathcal{O}_\varphi(P) = \emptyset$ then, for all but finitely many primes v,*

$$V(K_v) \cap \mathcal{C}_v(\mathcal{O}_\varphi(P)) = \emptyset.$$

Consequently, (X, V, φ) satisfies the strong dynamical Hasse principle.

The rest of this paper is organized as follows. In Section 4.3 we prove a local version of Theorem 4.2. Next in Section 4.4, we show how Theorem 4.2 follows from the local version, Theorem 4.4. In Section 4.5 we prove Theorem 4.3 and in Section 4.6 we give some closing remarks.

4.3. **The local statement.** Throughout this section, we work locally. Let A denote a complete discrete valuation ring, $\mathfrak{m}$ its maximal ideal, and k its residue field; we will assume that k is perfect. We write F for the fraction field of A.

The goal of this section is to prove the following:

Theorem 4.4. *Let $\mathscr{X}$ be a scheme of finite type over A, let φ be an étale endomorphism of $\mathscr{X}$, and let $\mathscr{V}$ be a reduced closed subscheme of $\mathscr{X}$ such that $\varphi^M(\mathscr{V}) \subseteq \mathscr{V}$ for some $M \geq 1$. Let $P \in \mathscr{X}(A)$. If $\mathscr{V}(A)$ does not intersect the orbit of P, then $\mathscr{V}(A)$ does not intersect the $\mathfrak{m}$-adic closure of the orbit of P.*

The current version of the proof follows the remarks by the referee, which have substantially simplified the original arguments. We shall also sketch an alternative proof in a somewhat more restrictive situation, based on p-adic uniformization of orbits ((Bell et al., 2010), rendered very elementary by a recent note of B. Poonen (Poonen, 2014)). This technique will also appear in the proof of Proposition 4.9.

We use the following very simple lemma:

Lemma 4.5. *Let R be a Noetherian ring and I an ideal of R. If φ is a ring automorphism of R such that $\varphi(I) \subseteq I$, then $\varphi(I) = I$.*

Proof. We assume that $\varphi(I) \subsetneq I$. Then

$$I \subsetneq \varphi^{-1}(I) \subsetneq \varphi^{-2}(I) \dots$$

violates the ascending chain condition. □

Proof of Theorem 4.4. Let $\bar{P}$ denote the image of P under the reduction map $\mathscr{X}(A) \to \mathscr{X}(k)$. If $\bar{P}$ is not preperiodic, then the orbit $\mathcal{O}_\varphi(P)$ equals its $\mathfrak{m}$-adic closure and we are done. Hence we may assume that $\bar{P}$ is preperiodic. By replacing P by a point in its orbit and replacing φ by an iterate, we may reduce to the case that $\varphi(\bar{P}) = \bar{P}$ and $\varphi(\mathscr{V}) \subseteq \mathscr{V}$. We may also assume $\bar{P} \in \mathscr{V}(k)$; otherwise the conclusion of the theorem is obvious.

Since P is not in $\mathscr{V}(A)$, we can choose n such that the image of P under $\mathscr{X}(A) \to \mathscr{X}(A/\mathfrak{m}^n)$ is not in $\mathscr{V}(A/\mathfrak{m}^n)$. The infinitesimal lifting property for the formally étale morphism φ shows that the diagram of sets:

$$\begin{tikzcd} \mathscr{X}(A/\mathfrak{m}^n) \arrow[r] \arrow[d, "\varphi"] & \mathscr{X}(k) \arrow[d, "\varphi"] \\ \mathscr{X}(A/\mathfrak{m}^n) \arrow[r] & \mathscr{X}(k) \end{tikzcd} \tag{4.3}$$

is Cartesian. Let T_1 be the set of preimages of $\bar{P}$ under the reduction map $\mathscr{X}(A/\mathfrak{m}^n) \to \mathscr{X}(k)$. We have that φ induces a permutation on T_1.

Let T_2 denote the set of preimages of $\bar{P}$ under the reduction map $\mathscr{V}(A/\mathfrak{m}^n) \to \mathscr{V}(k)$. Let $\hat{\mathcal{O}}_{\mathscr{X},\bar{P}}$ and $\hat{\mathcal{O}}_{\mathscr{V},\bar{P}}$ denote respectively the completion of the local rings $\mathcal{O}_{\mathscr{X},\bar{P}}$ and $\mathcal{O}_{\mathscr{V},\bar{P}}$ with respect to their maximal ideals. Let J denote the ideal of $\mathcal{O}_{\mathscr{X},\bar{P}}$ defining $\mathscr{V}$. Since φ is étale, it induces an automorphism $\hat{\varphi}: \hat{\mathcal{O}}_{\mathscr{X},\bar{P}} \to \hat{\mathcal{O}}_{\mathscr{X},\bar{P}}$. From Lemma 4.5 applied to the ideal J, using the assumption $\varphi(\mathscr{V}) \subseteq \mathscr{V}$, we deduce that $\hat{\varphi}$ induces an automorphism of $\hat{\mathcal{O}}_{\mathscr{V},\bar{P}}$. Hence φ induces a permutation on T_2.

For $m \geq 0$, we can use induction on m to show that the image of $\varphi^m(P)$ in $\mathscr{X}(A/\mathfrak{m}^n)$ is contained in $T_1 \setminus T_2$. Hence the image in $\mathscr{X}(A/\mathfrak{m}^n)$ of the orbit of P is disjoint from $\mathscr{V}(A/\mathfrak{m}^n)$. This finishes the proof of the theorem. □

For the rest of this section, we briefly explain another proof of Theorem 4.4 using an analytic uniformization of $\mathcal{O}_\varphi(P)$ as in (Bell et al., 2010) or (Amerik, 2011). This proof requires the *extra assumption* that F and A are finite extensions of $\mathbb{Q}_p$ and $\mathbb{Z}_p$, respectively, and $\mathscr{X}$ is smooth over A.

As before, we may assume that $\varphi(\bar{P}) = \bar{P}$ and $\varphi(\mathscr{V}) \subseteq \mathscr{V}$. Let $C(P)$ denote the set of points in $\mathscr{X}(A)$ whose image in $\mathscr{X}(k)$ is $\bar{P}$. Let g denote the Krull dimension of $\mathcal{O}_{\mathscr{X},\bar{P}}$. As explained in (Bell et al., 2010), there is an isomorphism of A-algebras:

$$\tau\colon \hat{\mathcal{O}}_{\mathscr{X},\bar{P}} \xrightarrow{\sim} A[[T_1, \dots, T_g]]$$

inducing a corresponding $\mathfrak{m}$-adic analytic homeomorphism

$$h_\tau : \ C(P) \cong \mathfrak{m}^g$$

mapping P to the origin $(0, \ldots, 0) \in \mathfrak{m}^g$.

Definition 4.6. *Identify $\hat{\mathcal{O}}_{\mathscr{X},\bar{P}}$ with $A[[T_1, \ldots, T_g]]$, and $C(P)$ with $\mathfrak{m}^g$ as above. We say that the orbit $\mathcal{O}_\varphi(P)$ has a* uniformization *if there exist power series $G_1, \ldots, G_g$ in $F[[T]]$ convergent on A such that:*

(i) $(G_1(0), \ldots, G_g(0)) = P$, *which equals* 0 *in* $\mathfrak{m}^g$, *and*
(ii) $\varphi(G_1(z), \ldots, G_g(z)) = (G_1(z+1), \ldots, G_g(z+1))$ *for all* z *in* $\mathbb{Z}_p$.

Proof of Theorem 4.4 using uniformization. By (Poonen, 2014) (which has simplified and generalized (Bell et al., 2010, Theorem 3.3) and (Amerik, 2011, Theorem 7)), there is a uniformization of $\mathscr{O}_\varphi(P)$ (possibly after replacing φ by an iterate). Let $G = (G_1, \ldots, G_g)$ be such a uniformization. By Definition 4.6, the $\mathfrak{m}$-adic closure of $\mathcal{O}_\varphi(P)$ is contained in $G(\mathbb{Z}_p)$. If there is some $u \in \mathbb{Z}_p$ such that $G(u) \in \mathscr{V}(A)$, then $G(u+n) \in \mathscr{V}(A)$ for every natural number n. Let $H = 0$ be any of the equations defining $\mathscr{V}$ in $\mathscr{X}$. Then we have $H \circ G(u+n) = 0$ for every natural number n. Since a nonzero p-adic analytic function on $\mathbb{Z}_p$ can have only finitely many zeros, the analytic function $H \circ G$ must be identically zero on $\mathbb{Z}_p$. Therefore $G(\mathbb{Z}_p) \subseteq \mathscr{V}(A)$ and so the whole orbit of P is contained in $\mathscr{V}(A)$, contradicting our assumption that $\mathscr{V}(A) \cap \mathcal{O}_\varphi(P) = \emptyset$. $\square$

4.4. **Proof of Theorem 4.2.** In this section, we present the proof of Theorem 4.2. First we show that for all but finitely many places v, the assumptions of Theorem 4.4 hold.

Lemma 4.7. *Assume that φ is étale and that $\varphi^m(V) \subseteq V$ for some $m \geq 1$. Fix a point $P \in X(K)$. Then there exists a finite set $S \subseteq M_K$ containing all the archimedean places such that X, V, and φ extend to models $\mathscr{X}$, $\mathscr{V}$, and $\tilde{\varphi}$, respectively, over $\mathcal{O}_{K,S}$ and P extends to $\mathscr{P} \in \mathscr{X}(\mathcal{O}_{K,S})$ with the following properties:*

- *$\mathscr{X}$ is quasi-projective over $\mathcal{O}_{K,S}$, $\tilde{\varphi}$ is étale,*
- *$\mathscr{V}$ is a reduced closed subscheme of $\mathscr{X}$ and $\tilde{\varphi}^m(\mathscr{V}) \subseteq \mathscr{V}$.*

Proof. Since X is quasi-projective over K, we can find a quasi-projective model $\mathscr{X}$ for X, models $\mathscr{V}$ and $\tilde{\varphi}$ for V and φ, respectively, over some $\mathcal{O}_{K,S}$. By enlarging S, we may assume P extends (uniquely since $\mathscr{X}$ is separated) to $\mathscr{P} \in \mathscr{X}(\mathcal{O}_{K,S})$. As the locus in $\mathrm{Spec}(\mathcal{O}_{K,S})$ over which $\mathscr{X}$ is not smooth, $\mathscr{V}$ is not flat, or $\tilde{\varphi}$ is not étale is closed, by enlarging S, we may assume that this locus is empty. Since $\mathscr{V}$ is flat over $\mathcal{O}_{K,S}$ and its generic fiber is reduced, $\mathscr{V}$ is itself reduced.

It remains to ensure that $\tilde{\varphi}^m(\mathscr{V}) \subseteq \mathscr{V}$. By enlarging S again, we may assume that every irreducible component of $\mathscr{V}$ contains some point in the generic fiber. Then since $\varphi^m(V) \subseteq V$ and V is dense in $\mathscr{V}$, we have $\tilde{\varphi}^m(\mathscr{V}) \subseteq \mathscr{V}$. $\square$

Proof of Theorem 4.2. Fix $S \subseteq M_K$ as in Lemma 4.7. Fix $v \notin S$. Now we apply Theorem 4.4 to have that $\mathscr{V}(\mathcal{O}_v)$ does not intersect the closure of $\mathcal{O}_{\tilde{\varphi}}(\mathscr{P})$ in $\mathscr{X}(\mathcal{O}_v)$. Since $\mathscr{X}(\mathcal{O}_v)$ is closed in $\mathscr{X}(K_v) = X(K_v)$, the set $V(K_v) = \mathscr{V}(K_v)$ does not intersect the closure of $\mathcal{O}_\varphi(P)$ in $X(K_v)$. $\square$

4.5. **Proof of Theorem 4.3.** We need the following useful result which might also be of independent interest.

Lemma 4.8. *Let X and φ be as in Theorem 4.3. Let Y be a closed irreducible φ-preperiodic subvariety of X and let Y_1 be a periodic iterate of Y. We recall from the introduction that this means there exist integers $k_0 \geq 0$ and $k > 0$ such that $Y_1 = \varphi^{k_0}(Y)$ and $\varphi^k(Y_1) = Y_1$. Then every irreducible component of $Y \cap Y_1$ is preperiodic.*

Proof. We may assume that $\varphi(Y) \neq Y$. Replacing φ by an iterate, we reduce to the case $Y_1 = \varphi(Y) = \varphi^2(Y) \neq Y$. Let $\nu_1 \colon \tilde{Y_1} \to Y_1$ and $\nu \colon \widetilde{\varphi^{-1}(Y_1)} \to \varphi^{-1}(Y_1)$ denote the normalizations of Y_1 and $\varphi^{-1}(Y_1)$, respectively. For every integer $n \geq 1$, define:

$$Z_n := \left\{x \in Y_1 \colon \ |\nu_1^{-1}(x)| \geq n\right\},$$

where $|\nu_1^{-1}(x)|$ is counted with multiplicity. By the semicontinuity theorem, Z_n is closed in Y_1.

Now let W be an irreducible component of $Y \cap Y_1$. Let $d = \dim W$ and let w be the generic point of W. Let $s = |\nu_1^{-1}(\varphi(w))|$. Since φ is closed, $\varphi(W)$ is the closure of $\{\varphi(w)\}$, which is contained in Z_s. We will prove that $\varphi(W)$ is an irreducible component of Z_s. Assume otherwise and let C_1 be an irreducible component of Z_s strictly containing $\varphi(W)$. Let c_1 denote the generic point of C_1. Since $c_1 \in Z_s$, we have $|\nu_1^{-1}(c_1)| \geq s$. On the other hand, by semicontinuity $|\nu_1^{-1}(c_1)| \leq |\nu_1^{-1}(\varphi(w))| = s$. Hence we must have $|\nu_1^{-1}(c_1)| = s$.

The étale morphism φ induces an étale morphism $\varphi^{-1}(Y_1) \to Y_1$. In particular, the induced morphism $\mathrm{Spec}(\mathcal{O}_{\varphi^{-1}(Y_1),w}) \to \mathrm{Spec}(\mathcal{O}_{Y_1,\varphi(w)})$ is flat, and hence, surjective by Exercise 10 and Exercise 11 in (Atiyah and Macdonald, 1969, p.68). Therefore, there exists c in $\varphi^{-1}(Y_1)$ such that $\varphi(c) = c_1$ and the Zariski closure C of c in $\varphi^{-1}(Y_1)$ contains W. Since taking normalization commutes with étale base change, we have a Cartesian diagram:

$$\begin{array}{ccc} \widetilde{\varphi^{-1}(Y_1)} & \longrightarrow & \tilde{Y_1} \\ \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \nu_1} \\ \varphi^{-1}(Y_1) & \longrightarrow & Y_1 \end{array}$$

This implies:

$$|\nu^{-1}(w)| = |\nu_1^{-1}(\varphi(w))| = s = |\nu_1^{-1}(c_1)| = |\nu^{-1}(c)| \tag{4.4}$$

Let $\mathscr{I}$ denote the set of irreducible components of $\varphi^{-1}(Y_1)$. Let $\mathscr{I}_w$ and $\mathscr{I}_c$ denote the set of irreducible components of $\varphi^{-1}(Y_1)$ containing w and c, respectively. Since $W \subseteq C$, we obviously have $\mathscr{I}_c \subseteq \mathscr{I}_w$. By comparing dimensions, we have that Y and Y_1 belong to $\mathscr{I}_w$. Since C_1 strictly contains $\varphi(W)$, we have that C strictly contains W. Note that it is impossible for both Y and Y_1 to contain C since W is an irreducible component of $Y \cap Y_1$ and C strictly contains W. Therefore either $Y \notin \mathscr{I}_c$ or $Y_1 \notin \mathscr{I}_c$. This gives that $\mathscr{I}_c$ is strictly contained in $\mathscr{I}_w$.

For each $T \in \mathscr{I}$, let $\nu_T : \ \tilde{T} \to T$ be the normalization of T. After composing with the embedding $T \hookrightarrow \varphi^{-1}(Y_1)$, we still use ν_T to denote the map $\tilde{T} \to \varphi^{-1}(Y_1)$. By definition of normalization, we may identify $\widetilde{\varphi^{-1}(Y_1)}$ with the disjoint union of $\tilde{T}$ for $T \in \mathscr{I}$. The morphism ν is induced from the morphisms ν_T for $T \in \mathscr{I}$. We have:

$$\begin{aligned} |\nu^{-1}(c)| &= \sum_{T \in \mathscr{I}_c} |\nu_T^{-1}(c)| \\ |\nu^{-1}(w)| &= \sum_{T \in \mathscr{I}_w} |\nu_T^{-1}(w)| \end{aligned} \tag{4.5}$$

By semicontinuity, we have $|\nu_T^{-1}(c)| \leq |\nu_T^{-1}(w)|$ for every $T \in \mathscr{I}_c$. Since $\mathscr{I}_w$ strictly contains $\mathscr{I}_c$, we have that $|\nu^{-1}(c)| < |\nu^{-1}(w)|$, contradicting (4.4). This proves that $\varphi(W)$ is an irreducible component of Z_s.

Now let $\mathcal{Z}$ denote the union of all d-dimensional irreducible components of all the Z_n's for $n \geq 1$. This is a finite union since $Z_n = \emptyset$ for all sufficiently large n thanks to finiteness of ν_1. We have proved that $\varphi(W)$ is an irreducible component of $\mathcal{Z}$. For each $m \geq 1$, repeating the same arguments for φ^m instead of φ, we have that $\varphi^m(W)$ is an irreducible component of $\mathcal{Z}$. This proves that W is preperiodic. □

We now begin the proof of Theorem 4.3. We may assume that V is irreducible. If V is periodic, we are done by Theorem 4.2. So assume that V is not periodic. Replacing φ by an iterate, we may assume $V_1 := \varphi(V) = \varphi^2(V) \neq V$.

Assume first that $V_1(K) \cap \mathcal{O}_\varphi(P) = \emptyset$. Then by Theorem 4.2, there is a finite set of places S_1 such that $V_1(K_v) \cap \mathcal{C}_v(\mathcal{O}_\varphi(P)) = \emptyset$ for $v \notin S_1$. For any such v, since φ maps $V(K_v) \cap \mathcal{C}_v(\mathcal{O}_\varphi(P))$ into $V_1(K_v) \cap \mathcal{C}_v(\mathcal{O}_\varphi(P))$, we have $V(K_v) \cap \mathcal{C}_v(\mathcal{O}_\varphi(P)) = \emptyset$ too. Thus the conclusion of Theorem 4.3 holds.

Now assume that $\varphi^m(P) \in V_1(K)$ for some $m \geq 0$. For every v, we have

$$\mathcal{C}_v(\mathcal{O}_\varphi(\varphi^m(P))) \subseteq V_1(K_v) \tag{4.6}$$

$$\mathcal{C}_v(\mathcal{O}_\varphi(P)) = \mathcal{C}_v(\mathcal{O}_\varphi(\varphi^m(P))) \cup \left\{P, \ldots, \varphi^{m-1}(P)\right\}. \tag{4.7}$$

Assume that $V(K_v) \cap \mathcal{C}_v(\mathcal{O}_\varphi(P)) \neq \emptyset$. Because of (4.7) and $V(K) \cap \mathcal{O}_\varphi(P) = \emptyset$, we have that $V(K_v) \cap \mathcal{C}_v(\mathcal{O}_\varphi(\varphi^m(P))) \neq \emptyset$. Hence (4.6) implies:

$$(V \cap V_1)(K_v) \cap \mathcal{C}_v(\mathcal{O}_\varphi(P)) \neq \emptyset.$$

This can only happen for finitely many v's by the induction hypothesis applied to $V \cap V_1$, whose irreducible components are preperiodic by Lemma 4.8.

4.6. **Closing remarks.** It is natural to ask whether the assumption in Theorem 4.2 that V is preperiodic is necessary. In the next proposition, let K be a number field, let X be a quasi-projective variety over K, let $\varphi : X \to X$ be an étale K-endomorphism, and let $V = Q \in X(K)$ be a single point. For simplicity, we assume that X is smooth over K (see Remark 4.10). Fix a finite set of places S of K containing all the archimedean ones, a smooth quasi-projective model $\mathscr{X}$ of X over $\mathcal{O}_{K,S}$, an extension $\tilde{\varphi}$ of φ such that $\tilde{\varphi}$ is an étale $\mathcal{O}_{K,S}$-endomorphism of $\mathscr{X}$, and an extension $\mathscr{Q} \in \mathscr{X}(\mathcal{O}_{K,S})$ of Q. We say that $\mathscr{Q}$ has almost everywhere periodic reduction if for all but finitely many primes $\mathfrak{p}$, the reduction of $\mathscr{Q}$ modulo $\mathfrak{p}$ is periodic. The next result shows that (X, Q, φ) fails the strong dynamical Hasse principle if and only if Q is not periodic but $\mathscr{Q}$ has almost everywhere periodic reduction.

Proposition 4.9. *Let K, X, φ, S, $\mathscr{X}$ and Q be as in the previous paragraph.*

(a) *For all primes $\mathfrak{p} \notin S$, the following is true: if the reduction of $\mathscr{Q}$ modulo $\mathfrak{p}$ is periodic then the $\mathfrak{p}$-adic closure of the orbit of $\varphi(Q)$ contains Q. Consequently, if Q is not periodic but $\mathscr{Q}$ has almost everywhere periodic reduction then (X, Q, φ) does not satisfy the strong Hasse principle.*

(b) *Conversely, if Q is either periodic or $\mathscr{Q}$ does not have almost everywhere periodic reduction, then (X, Q, φ) satisfies the strong dynamical Hasse principle.*

Proof. (a) The first assertion follows immediately from the $\mathfrak{p}$-adic uniformization of the φ^N-orbit of Q (for some integer N, sufficiently large depending on $\mathfrak{p}$) and the fact that for every analytic function G from $\mathcal{O}_{\mathfrak{p}}$ to $\mathcal{O}_{\mathfrak{p}}^g$, the point $G(0)$ lies in the closure of $\{G(1), G(2), \ldots\}$ (see e.g. (Poonen, 2014), cf. Subsection 4.2, for the existence of the uniformization). For the second assertion, note that the orbit of $P = \varphi(Q)$ does not contain Q but the $\mathfrak{p}$-adic closure of this orbit contains Q by the first assertion.

(b) The case that Q is periodic follows from Theorem 4.2. Hence we assume that Q is non-periodic. Let $P \in X(K)$ be such that $Q \notin \mathcal{O}_\varphi(P)$. Let $\mathfrak{p} \notin S$ be such that P extends to a point $\mathscr{P} \in \mathscr{X}(\mathcal{O}_{\mathfrak{p}})$. If $Q \in \mathcal{C}_{\mathfrak{p}}(\mathcal{O}_\varphi(P))$ then there exist infinitely many m such that $\mathscr{Q}$ and $\tilde{\varphi}^m(\mathscr{P})$ have the same reduction modulo $\mathfrak{p}$, so this implies that $\mathscr{Q}$ is periodic modulo $\mathfrak{p}$. But there are infinitely many primes $\mathfrak{p}$ such that this conclusion does not hold, thanks to our assumption on Q; hence (X, Q, φ) satisfies the strong dynamical Hasse principle. $\square$

Remark 4.10. For simplicity, we assumed that X is smooth over K in Proposition 4.9. It is not difficult to remove this assumption by using the fact that étale morphisms preserve the smooth locus. In other words, let X' be the smooth locus of X over K and let $X'' = X - X'$; then we have that φ induces étale self-maps on X' and X''. By enlarging S and taking closure in $\mathscr{X}$, we have a model of X'' over $\mathcal{O}_{K,S}$ which is the non-smooth locus of $\mathscr{X}$ and φ extends to an étale self-map of this model over $\mathcal{O}_{K,S}$. We now take the complement in $\mathscr{X}$ of the above model to obtain a model of X'. Enlarging S further if necessary, we may assume that φ extends to an étale sell-map of the above model of X'. The proof of Proposition 4.9 settles the case $Q \in X'(K)$. If $Q \in X''(K)$, we can use Noetherian induction since X'' is a strictly smaller closed subvariety of X. This kind of argument has appeared in (Bell et al., 2010).

Some results in the literature suggest that the examples of non-periodic points with almost everywhere periodic reduction must be very special, and so the strong dynamical Hasse principle mostly holds when the endomorphism is étale and the subvariety is a single point (but we remark that, on the contrary, if the dimension of the subvariety is large, the heuristics in Section 2 indicate the failure of the strong dynamical Hasse principle). For instance, by a result of Pink (Pink, 2004, Corollary 4.3), such points cannot exist for the multiplication-by-d map on an abelian variety. Furthermore, by (Benedetto et al., 2013, Corollary 1.2), such points also cannot exist for a self-map of $\mathbb{P}^1$ of degree at least two (though such a map is not étale, and thus Proposition 4.9 does not apply directly). On the other hand, such points exist for automorphisms of infinite order (eventually after a finite extension of the base field). It seems reasonable to conjecture that non-periodic points with almost everywhere periodic reduction do not exist for polarized morphisms φ (that is, morphisms such that $\varphi^*\mathcal{L} = \mathcal{L}^{\otimes k}$ for some integer $k > 1$ and some ample line bundle $\mathcal{L}$), so that the strong dynamical Hasse principle holds for number fields K, étale polarized morphisms φ and $V \in X(K)$. Notice however that étale polarized endomorphisms are extremely rare, cf. (Fakhruddin, 2003).

We conclude this paper by proving that for curves over number fields, the only counterexamples to the dynamical Brauer-Manin criterion are automorphisms φ of a very special kind.

Proposition 4.11. *Let X be a smooth geometrically integral projective curve of genus g over a number field K, let φ be a nonconstant self-map of X over K, and let V be a finite subset of $X(K)$. We make the following additional assumptions.*

(a) *If $X = \mathbb{P}^1$, assume that φ is not conjugate to $z \mapsto z+1$.*
(b) *If $g = 1$, assume that φ has a preperiodic point in $X(\bar{K})$. (If we regard X as an elliptic curve, this condition is equivalent to the condition that φ is not a translate by a non-torsion point.)*

If $P \in X(K)$, then:

$$V(K) \cap \mathcal{O}_\varphi(P) = V(K,S) \cap \mathcal{C}(\mathcal{O}_\varphi(P)). \tag{4.8}$$

Remark 4.12. When X is an abelian variety, V is an arbitrary subvariety and φ is a K-endomorphism of X such that $\mathbb{Z}[\varphi]$ is an integral domain, Hsia and Silverman show that (4.8) holds under certain strong conditions. We refer the readers to (Hsia and Silverman, 2009, Theorem 11) for more details. Our proof of Proposition 4.11 gives an unconditional proof of their result when X is an elliptic curve.

Proof of Proposition 4.11. The case $g \geq 2$ is trivial since all endomorphisms of curves of genus at least two are of finite order. The case $X = \mathbb{P}^1$ has been settled by Silverman and Voloch (see Theorem 1 and Remark 9 in (Silverman and Voloch, 2009)).

Now consider the case when $g = 1$. The case that P is φ-preperiodic is easy, so we assume that P is not preperiodic. There exists a non-negative integer N such that $\varphi^M(P) \notin V(K)$ for all $M > N$. After replacing P by $\varphi^{N+1}(P)$, we may assume the φ-orbit of P does not intersect $V(K)$. It remains to show that $V(K,S) \cap \mathcal{C}(\mathcal{O}_\varphi(P)) = \emptyset$. By assumption (b), there is some $M > 0$ such that φ^M has a fixed point. By replacing the data (φ, P) with $(\varphi^M, \varphi^i(P))$ for $0 \leq i < M$, we may assume that φ has a fixed point.

Note that if we can prove

$$V(L, S_L) \cap \mathcal{C}(\mathcal{O}_\varphi(P)) = \emptyset$$

for L a finite extension of K, and $S_L \subset M_L$ the set of places of L lying above places in S, then this implies that $V(K,S) \cap \mathcal{C}(\mathcal{O}_\varphi(P)) = \emptyset$. Thus, we may assume that φ has a fixed point $O \in X(K)$. By Theorem 4.2, for all but finitely many primes $\mathfrak{p}$ of K, we have $\{O\} \cap \mathcal{C}_\mathfrak{p}(\mathcal{O}_\varphi(P)) = \emptyset$. We may enlarge S (and by abuse of notation) to assume that X is an elliptic curve over $\mathcal{O}_{K,S}$ with identity $O \in X(\mathcal{O}_{K,S})$, φ is an endomorphism of X over $\mathcal{O}_{K,S}$, $P \in X(\mathcal{O}_{K,S})$ is not preperiodic under φ, $\{O\} \cap \mathcal{C}_\mathfrak{p}(\mathcal{O}_\varphi(P)) = \emptyset$ for every $\mathfrak{p} \notin S$, and $V(K) \setminus \{O\} \subseteq (X \setminus \{O\})(\mathcal{O}_{K,S})$.

By Siegel's theorem, $(X - O)(\mathcal{O}_{K,S})$ is finite, so some iterate P' of P is not in it. Hence there exists some $\mathfrak{p} \notin S$ such that P' reduces to O modulo $\mathfrak{p}$. Then all iterates of P' reduce to O modulo $\mathfrak{p}$ since $\varphi(O) = O$. Thus $\mathcal{C}_\mathfrak{p}(\mathcal{O}_\varphi(P))$ cannot contain any point of $V(K) - \{O\} = V(K_\mathfrak{p}) - \{O\}$. Together with the condition $\{O\} \cap \mathcal{C}_\mathfrak{p}(\mathcal{O}_\varphi(P)) = \emptyset$, we have that $V(K) \cap \mathcal{C}_\mathfrak{p}(\mathcal{O}_\varphi(P)) = \emptyset$. So $V(K,S) \cap \mathcal{C}(\mathcal{O}_\varphi(P)) = \emptyset$. This finishes the proof. $\square$

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