VARIATION OF THE NAZAROV-SODIN CONSTANT FOR RANDOM PLANE WAVES AND ARITHMETIC RANDOM WAVES

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ABSTRACT. This is a manuscript containing the full proofs of results announced in [KW], together with some recent updates. We prove that the *Nazarov-Sodin constant*, which up to a natural scaling gives the leading order growth for the expected number of nodal components of a random Gaussian field, genuinely depends on the field. We then infer the same for "arithmetic random waves", i.e. random toral Laplace eigenfunctions.

1. INTRODUCTION

For $m \geq 2$, let $f : \mathbb{R}^m \to \mathbb{R}$ be a *stationary* centred Gaussian random field, and $r_f : \mathbb{R}^m \to \mathbb{R}$ its covariance function

$$r_f(x) = \mathbb{E}[f(0) \cdot f(x)] = \mathbb{E}[f(y) \cdot f(x+y)].$$

Given such an f, let ρ denote its spectral measure, i.e. the Fourier transform of r_f (assumed to be a probability measure); note that prescribing ρ uniquely defines f by Kolmogorov's Theorem (cf. [CL, Chapter 3.3].) In what follows we often allow for ρ to vary; it will be convenient to let f_ρ denote a random field with spectral measure ρ. We further assume that a.s. f_ρ is sufficiently smooth, and that the distribution of f and its derivatives is non-degenerate in an appropriate sense (a condition on the support of ρ).

1.1. Nodal components and the Nazarov-Sodin constant.

Let $\mathcal{N}(f_\rho; R)$ be the number of connected components of $f_\rho^{-1}(0)$ in $B_0(R)$ (the radius-R ball centred at 0), usually referred to as the *nodal components* of f_ρ; $\mathcal{N}(f_\rho; R)$ is a random variable. Assuming further that f_ρ is ergodic (equivalently, ρ has no atoms), with non-degenerate gradient distribution (equivalent to ρ not being supported on a hyperplane passing through the origin), Nazarov and Sodin [So, Theorem 1] evaluated the expected number of nodal components of f_ρ to be asymptotic to

$$\mathbb{E}[\mathcal{N}(f_\rho; R)] = c_{NS}(\rho) \cdot \mathrm{Vol}(B(1)) \cdot R^m + o(R^m), \tag{1.1}$$

where $c_{NS}(\rho) \geq 0$ is a constant, subsequently referred to as the "Nazarov-Sodin constant" of f_ρ, and $\mathrm{Vol}(B(1))$ is the volume of the radius-1 m-ball $B(1) \subseteq \mathbb{R}^m$. They also established convergence in mean, i.e., that

$$\mathbb{E}\left[\left|\frac{\mathcal{N}(f_\rho; R)}{\mathrm{Vol}(B(1)) \cdot R^m} - c_{NS}(\rho)\right|\right] \to 0; \tag{1.2}$$

it is a consequence of the assumed ergodicity of the underlying random field f_ρ. In this manuscript we will consider c_{NS} as a function of the spectral density ρ, without assuming that f_ρ is ergodic. To our best knowledge, the value of $c_{NS}(\rho)$, even for a single ρ, was not rigorously known heretofore.

For $m = 2$, $\rho = \rho_{\mathcal{S}^1}$ the uniform measure on the unit circle $\mathcal{S}^1 \subseteq \mathbb{R}^2$ (i.e. $d\rho = \frac{d\theta}{2\pi}$ on $\mathcal{S}^1$, and vanishing outside the circle) the corresponding random field f_{RWM} is known as *random monochromatic wave*; according to Berry's *Random Wave Model* [Be], f_{RWM} serves as a universal model for Laplace eigenfunctions on generic surfaces in the high energy limit. The corresponding *universal* Nazarov-Sodin constant

$$c_{\text{RWM}} = c_{NS}\left(\frac{d\theta}{2\pi}\right) > 0 \tag{1.3}$$

was proven to be strictly positive [NS2]. Already in [BS], Bogomolny and Schmit employed the percolation theory to predict its value, but recent numerics by Nastacescu [Na], Konrad [Ko] and Beliaev-Kereta [BK], consistently indicate a $4.5-6\%$ deviation from these predictions.

More generally, let $(\mathcal{M}^m, g)$ be a smooth compact Riemannian manifold of volume $\text{Vol}(\mathcal{M})$. Here the restriction of a fixed random field $f : \mathcal{M} \to \mathbb{R}$ to growing domains, as was considered on the Euclidean space, makes no sense. Instead we consider a sequence of smooth non-degenerate random fields $\{f_L\}_{L\in\mathcal{L}}$ (for $\mathcal{L} \subseteq \mathbb{R}$ some discrete subset), and the total number $\mathcal{N}(f_L)$ of nodal components of f_L on $\mathcal{M}$ (the case $\mathcal{M} = \mathbb{T}^2$ will be treated in § 1.3; $\mathcal{L}$ will then be a subset of the Laplace spectrum for $\mathbb{T}^2$). Here we may define a scaled covariance function of f_L around a fixed point $x \in \mathcal{M}$ on its tangent space $T_x(\mathcal{M}) \cong \mathbb{R}^m$ via the exponential map at x, and assume that for a.e. $x \in \mathcal{M}$ the scaled covariance and a few of its derivatives converge, locally uniformly, to the covariance function of a limiting stationary Gaussian field around x and its respective derivatives; let ρ_x be the corresponding spectral density. For the setup as above, Nazarov-Sodin proved [So, Theorem 4] that as $L \to \infty$,

$$\mathbb{E}[\mathcal{N}(f_L)] = \overline{c_{NS}} \cdot \text{Vol}(\mathcal{M}) \cdot L^m + o(L^m),$$

for some $\overline{c_{NS}} \geq 0$ depending on the limiting fields only, or, more precisely,

$$\overline{c_{NS}} = \int\limits_{\mathcal{M}} c_{NS}(\rho_x)dx$$

is the superposition of their Nazarov-Sodin constants. This result applies in particular to random band-limited functions on a generic Riemannian manifold, considered in [SW], with the constant $\overline{c_{NS}} > 0$ strictly positive.

1.2. **Statement of results for random waves on $\mathbb{R}^2$.** Let $\mathcal{P}$ be the collection of probability measures on $\mathbb{R}^2$ supported on the radius-1 standard ball $B(1) \subseteq \mathbb{R}^2$, and invariant under rotation by π. We note that any spectral measure can without loss of generality be assumed to be π-rotation invariant, hence the collection of spectral measures supported on $B(R)$ can, after rescaling, be assumed to lie in $\mathcal{P}$.

Our first goal (Proposition 1.1 below) is to extend the definition of the Nazarov-Sodin constant for all $\rho \in \mathcal{P}$, in particular, we allow spectral measures possessing atoms. We show that one may define c_{NS} on $\mathcal{P}$ such that the defining property (1.1) of c_{NS} is satisfied, though its stronger form (1.2) might not necessarily hold. Further, the limit on the l.h.s. of (1.2) always exists, even if it is not vanishing (Proposition 1.2 below, cf. § 7). Rather than counting the nodal components lying in discs of increasing radius, we will count components lying in squares with increasing side lengths; by abuse of notation from now on $\mathcal{N}(f_\rho; R)$ will denote the number of nodal components of f_ρ lying in the square

$$\mathcal{D}_R := [-R, R]^2 \subseteq \mathbb{R}^2.$$

Though the results are equivalent for both settings (every result we are going to formulate on domains lying in squares could equivalently be formulated for discs), unlike discs, the squares possess the extra-convenience of tiling into smaller squares. This obstacle could be easily mended for the discs using the ingenious "Integral-Geometric Sandwich" (which can be viewed as an infinitesimal tiling) introduced by Nazarov-Sodin [So].

Proposition 1.1. *Let f_ρ be a plane random field with spectral density $\rho \in \mathcal{P}$. The limit*

$$c_{NS}(\rho) := \lim_{R\to\infty} \frac{\mathbb{E}[\mathcal{N}(f_\rho; R)]}{4R^2}$$

exists and is uniform w.r.t. $\rho \in \mathcal{P}$. More precisely, for every $\rho \in \mathcal{P}$ we have

$$\mathbb{E}[\mathcal{N}(f_\rho; R)] = c_{NS}(\rho) \cdot 4R^2 + O(R) \tag{1.4}$$

with constant involved in the 'O'-notation absolute.

As for fluctuations around the mean à la (1.2), we have the following result:

Proposition 1.2. *The limit*

$$d_{NS}(\rho) := \lim_{R\to\infty} \mathbb{E}\left[\left|\frac{\mathcal{N}(f_\rho; R)}{4R^2} - c_{NS}(\rho)\right|\right] \tag{1.5}$$

("Nazarov-Sodin discrepancy functional") exists for all $\rho \in \mathcal{P}$.

However, the limit (1.5) is not uniform w.r.t. $\rho \in \mathcal{P}$, so in particular, an analogue of (1.4) does not hold for $d_{NS}(\cdot)$. For had (1.5) been uniform, a proof similar to the proof of Theorem 1.3 below would yield the continuity of $d_{NS}(\cdot)$; this cannot hold[1], since on one hand it is possible to construct a measure $\rho \in \mathcal{P}$ with $d_{NS} > 0$ (see § 7), and on the other hand it is possible to approximate an arbitrary measure $\rho \in \mathcal{P}$ with a smooth one ρ' (e.g. by convolving with smooth mollifiers), so that $f_{\rho'}$ is ergodic, and $d_{NS}(\rho') = 0$.

We believe that the uniform rate of convergence (1.4) is of two-fold independent interest. First, for numerical simulations it determines the value of sufficiently big radius R to exhibit a realistic nodal portrait with the prescribed precision. Second, it is instrumental for the proof of Theorem 1.3 below, a principal result of this manuscript.

Theorem 1.3. *The map $c_{NS} : \mathcal{P} \to \mathbb{R}_{\geq 0}$, given by*

$$c_{NS} : \rho \mapsto c_{NS}(\rho)$$

is a continuous functional w.r.t. the weak- topology on $\mathcal{P}$.*

To prove Theorem 1.3 we follow the steps of Nazarov-Sodin [So] closely, controlling the various error terms encountered. One of the key aspects of our proof, different from Nazarov-Sodin's, is the uniform version (1.4) of (1.1).

Giving good lower bounds on $c_{NS}(\rho)$ appears difficult and it is not a priori clear that $c_{NS}(\rho)$ genuinely varies with ρ. However, it is straightforward to see that $c_{NS}(\rho) = 0$ if ρ is a delta measure supported at zero, and we can also construct examples of monochromatic random waves with $c_{NS}(\rho) = 0$ when ρ is supported on two antipodal points. (See § 1.3 for some further examples of measures ρ satisfying stronger symmetry assumptions, yet with the property that $c_{NS}(\rho) = 0$.) This, together with the convexity and compactness of $\mathcal{P}$, easily gives the following consequence of Theorem 1.3.

Corollary 1.4. *The Nazarov-Sodin constant $c_{NS}(\rho)$ for $\rho \in \mathcal{P}$ attains all values in an interval of the form $[0, c_{\max}]$ for some $0 < c_{\max} < \infty$.*

Corollary 1.4 sheds no light on the value of $c_{\max}$; see § 2 for some intuition and related conjectures.

1.3. **Statement of results for toral eigenfunctions (arithmetic random waves).** Let S be the set of all integers that admit a representation as a sum of two integer squares, and let $n \in S$. The toral Laplace eigenfunctions $f_n : \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}$ of eigenvalue $-4\pi^2 n$ may be expressed as

$$f_n(x) = \sum_{\substack{\|\lambda\|^2=n \\ \lambda\in\mathbb{Z}^2}} a_\lambda e^{2\pi i\langle x,\lambda\rangle} \tag{1.6}$$

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[1] We are grateful to Dmitry Beliaev for pointing it out to us

for some complex coefficients $\{a_\lambda\}_\lambda$ satisfying $a_{-\lambda} = \overline{a_\lambda}$. We endow the space of eigenfunctions with a Gaussian probability measure by making the coefficient a_λ i.i.d. standard Gaussian (save for the relation $a_{-\lambda} = \overline{a_\lambda}$).

For this model ("arithmetic random waves") it is known [KKW, RW] that various local properties of f_n, e.g. the length fluctuations of the nodal line $f_n^{-1}(0)$, the number of nodal intersections against a reference curve, or the number of nodal points with a given normal direction, depend on the limiting angular distribution of $\{\lambda \in \mathbb{Z}^2 : \|\lambda\|^2 = n\}$. For example, in [RW2] the nodal length fluctuations for generic eigenfunctions was shown to vanish (this can be viewed as a refinement of Yau's conjecture [Y, Y2]), and in [KKW] the leading order term of the variance of the fluctuations was shown to depend on the angular distribution of $\{\lambda \in \mathbb{Z}^2 : \|\lambda\|^2 = n\}$. To make the notion of angular distribution precise, for $n \in S$ let

$$\mu_n = \frac{1}{r_2(n)} \sum_{\|\lambda\|^2=n} \delta_{\lambda/\sqrt{n}}, \tag{1.7}$$

where δ_x is the Dirac delta at x, be a probability measure on the unit circle $\mathcal{S}^1 \subseteq \mathbb{R}^2$. It is then natural (or essential) to pass to subsequences $\{n_j\} \subseteq S$ such that μ_{n_j} converges to some μ in the weak-$*$ topology, a probability measure on $\mathcal{S}^1$, so that the various associated quantities, such as the nodal length variance $\mathrm{Var}(f_n^{-1}(0))$ exhibit an asymptotic law. In this situation we may identify μ as the spectral density of the limiting field around each point of the torus when the unit circle is considered embedded $\mathcal{S}^1 \subseteq \mathbb{R}^2$ (see Lemma 5.6); such a limiting probability measure μ necessarily lies in the set $\mathcal{P}_{\text{symm}}$ of "monochromatic" probability measures on $\mathcal{S}^1$, invariant w.r.t. $\pi/2$-rotation and complex conjugation (i.e. $(x_1, x_2) \mapsto (x_1, -x_2)$). In fact, the family of weak-* partial limits of $\{\mu_n\}$ ("attainable" measures) is known [KW2] to be a proper subset of $\mathcal{P}_{\text{symm}}$.

Let $\mathcal{N}(f_n)$ denote the total number of nodal components of f_n on $\mathbb{T}^2$. On one hand, an application of [So, Theorem 4] yields[2] that if, as above, $\mu_{n_j} \Rightarrow \mu$ for μ a probability measure on $\mathcal{S}^1$, we have

$$\mathbb{E}[\mathcal{N}(f_{n_j})] = c_{NS}(\mu) \cdot n_j + o(n_j), \tag{1.8}$$

with the leading constant $c_{NS}(\mu)$ same as for the scale-invariant model (1.1), cf. [Ro, Theorem 1.2]. On the other hand, we will be able to infer from Proposition 1.1 the more precise *uniform* statement (1.11), by considering f_n on the square $[0,1]^2$ via the natural quotient map $q : \mathbb{R}^2 \hookrightarrow \mathbb{T}^2$ (see the proof of Theorem 1.5 part 1).

For $\mu \in \mathcal{P}_{\text{symm}}$ we can classify all measures μ such that $c_{NS}(\mu) = 0$, in particular classify when the leading constant on the r.h.s. of (1.8) vanishes. Namely, for $\theta \in [0, 2\pi]$ let

$$z(\theta) := (\cos(\theta), \sin(\theta)) \in \mathcal{S}^1 \subseteq \mathbb{R}^2,$$

$$\nu_0 = \frac{1}{4} \sum_{k=0}^{3} \delta_{z(k\cdot\pi/2)} \tag{1.9}$$

be the *Cilleruelo* measure [Ci], and

$$\widetilde{\nu_0} = \frac{1}{4} \sum_{k=0}^{3} \delta_{z(\pi/4+k\cdot\pi/2)} \tag{1.10}$$

be the *tilted* Cilleruelo measure; these are the only measures in $\mathcal{P}_{\text{symm}}$ supported on precisely 4 points. In addition to the aforementioned classification of measures $\mu \in \mathcal{P}_{\text{symm}}$ with $c_{NS}(\mu) = 0$ we prove the following concerning the rate of convergence (1.8), and the range of possible constants $c_{NS}(\mu)$ appearing

[2] Considering c_{NS} in the more general sense as in Proposition 1.1, and making the necessary adjustments in case μ does not fall into the class of spectral measures considered by Nazarov-Sodin.

on the r.h.s. of (1.8). (Note that the Nazarov-Sodin constant on the r.h.s. of (1.11) is associated with μ_n as opposed to the r.h.s. of (1.8), which is associated with the limiting measure μ.)

Theorem 1.5. *(1) We have uniformly for $n \in S$*

$$\mathbb{E}[\mathcal{N}(f_n)] = c_{NS}(\mu_n) \cdot n + O(\sqrt{n}), \tag{1.11}$$

with the constant involved in the 'O'-notation absolute.

(2) If $\mu_{n_j} \Rightarrow \mu$ for some subsequence $\{n_j\} \subseteq S$, where μ has no atoms, then convergence in mean holds:

$$\mathbb{E}\left[\left|\frac{\mathcal{N}(f_{n_j})}{n_j} - c_{NS}(\mu)\right|\right] = o_\rho(1). \tag{1.12}$$

(3) For $\mu \in \mathcal{P}_{symm}$, $c_{NS}(\mu) = 0$ if and only if $\mu = \nu_0$ or $\mu = \widetilde{\nu_0}$.

(4) For μ in the family of weak- partial limits of $\{\mu_n\}$, the functional $c_{NS}(\mu)$ attains all values in an interval of the form $I_{NS} = [0, d_{max}]$ with some $d_{max} > 0$.*

It is opportune to mention that D. Beliaev has informed us that he, together with M. McAuley and S. Muirhead, recently obtained a full classification the set of measures $\rho \in \mathcal{P}$ for which $c_{NS}(\rho) = 0$.

1.4. **Acknowledgments.** It is a pleasure to thank M. Sodin for many stimulating and fruitful discussions, insightful and critical comments while conducting the research presented, and his comments on an earlier version of this manuscript. We would also like to thank Z. Rudnick for many fruitful discussions and his help in improving the present manuscript, D. Beliaev for his valuable comments, in particular regarding the Nazarov-Sodin discrepancy functional $d_{NS}(\cdot)$, J. Buckley and M. Krishnapur for many stimulating conversations, and S. Muirhead for pointing the Gaussian Correlation Inequality [Roy] to us, as well as other useful comments on an earlier version of this manuscript. Finally, we thank P. Sarnak and J. Bourgain for their interest in our work and their support, and the anonymous referee for his valuable comments that improved the readability of our manuscript.

P.K. was partially supported by grants from the Göran Gustafsson Foundation, and the Swedish Research Council (621-2011-5498 and 621-2016-03701.) I.W. was partially supported by the EPSRC under the First Grant scheme (EP/J004529/1). Further, the research leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013) / ERC grant agreement n^o 335141 (I.W.).

2. Discussion and outline of key ideas

2.1. **Continuity of the number of nodal domains.** Theorem 1.3, a principal result of this paper, states that the expected number $\mathbb{E}\left[\mathcal{N}(f_\rho; R)\right]$ of nodal domains of f_ρ lying in a compact domain of $\mathbb{R}^2$, properly normalized, is continuous in the limit $R \to \infty$, namely $c_{NS}(\rho)$. We believe that it is in fact continuous without taking the limit, i.e. for R *fixed*, the function

$$\rho \to \mathbb{E}[\mathcal{N}(f_\rho; R)]$$

is a continuous function on $\mathcal{P}$.

2.2. **Maximal Nazarov-Sodin constant.** As for the maximal possible value of c_{NS}, it seems reasonable to assume that, in order to maximize the nodal domains number for $\rho \in \mathcal{P}$, one had better maximize the weight of the highest possible wavenumber. That is, to attain $c_{\max}$ as in Corollary 1.4 the measure ρ should be supported on $\mathcal{S}^1 \subseteq \mathbb{R}^2$, i.e. the random wave f_ρ must be monochromatic. Among those measures $\rho \in \mathcal{P}$ supported in $\mathcal{S}^1$ we know that the more concentrated ones (i.e. those supported on two antipodal points, or, for $\rho \in \mathcal{P}_{\text{symm}}$, Cilleruelo measure ν_0 supported on 4 symmetric points $\pm 1, \pm i$) *minimize* the nodal domains number (Theorem 1.5, part 3); (tilted) Cilleruelo measure is known to minimize other local quantities [KKW] when the uniform measure maximizes it, or vice versa.

For example, it is easy to see that $\mathbb{E}[\mathcal{N}(f,R)]$ is bounded above by the expected number of points $x=(x_1,x_2)\in\mathcal{D}_R$ such that

$$f(x)=0=\frac{\partial f}{\partial x_1}(x)+\frac{\partial f}{\partial x_2}(x),$$

and this expectation can be shown to be minimal for the Cilleruelo measure. Now, as the upper bound expectation is not invariant with respect to change of coordinates via rotation it is natural to chose the optimal rotation. That is, given a spectral measure ρ one should optimize by choosing a rotation that minimizes the above upper bound. The Cilleruelo measure, as well as the twisted one, has a minimal optimized upper bound, whereas the uniform measure has a maximal optimized upper bound.

It thus seems plausible that the uniform measure $\rho=\frac{d\theta}{2\pi}$ on $\mathcal{S}^1$ corresponding to the Nazarov-Sodin constant c_{RWM} in (1.3) *maximizes* the Nazarov-Sodin constant; since it happens $\frac{d\theta}{2\pi}\in\mathcal{P}_{\text{symm}}$ to lie in $\mathcal{P}_{\text{symm}}$, and is also a weak-$*$ limit of $\{\mu_n\}$ in (1.7), it then also maximizes the Nazarov-Sodin constant restricted as in Theorem 1.5. The above discussion is our motivation for the following conjecture regarding the maximal possible values $c_{\max}$ (resp. $d_{\max}$) of the Nazarov-Sodin constant.

Conjecture 2.1. (1) For $\mu\in\mathcal{P}_{\text{symm}}$ that are weak-$*$ limits of $\{\mu_n\}$, the maximal value $d_{\max}$ is uniquely attained by $c_{NS}(\mu_{\mathcal{S}^1})$, where $\mu_{\mathcal{S}^1}$ is the uniform measure on $\mathcal{S}^1\subseteq\mathbb{R}^2$. In particular,

$$d_{\max}=c_{\text{RWM}}.$$

(2) For $\rho\in\mathcal{P}$, the maximal value $c_{\max}$ is uniquely attained by $c_{NS}(\rho)$ for ρ the uniform measure on $\mathcal{S}^1\subseteq\mathbb{R}^2$. In particular

$$c_{\max}=d_{\max}=c_{\text{RWM}}.$$

2.3. **Cilleruelo sequences for arithmetic random waves.** On one hand Theorem 1.5 shows that, if one stays away from the Cilleruelo measure, it is possible to infer the asymptotic behavior of the toral nodal domains number $\mathcal{N}(f_n)$ from the asymptotic behavior of $\mathcal{N}(f_\rho;\cdot)$ where $\rho=\mu_n$ is the spectral measure of f_n when considered on $\mathbb{R}^2$. On the other hand, if $\{n_j\}\subseteq S$ is a Cilleruelo sequence, i.e.,

$$\mu_{n_j}\Rightarrow\nu_0, \tag{2.1}$$

then from part 3 of Theorem 1.5 we can only infer that

$$\lim_{j\to\infty}\frac{\mathbb{E}[\mathcal{N}(f_{n_j})]}{n_j}=0,$$

with no further understanding of the true asymptotic behavior of $\mathbb{E}[\mathcal{N}(f_{n_j})]$.

It is possible to realize the Euclidean random field $f_{\nu_0}:\mathbb{R}^2\to\mathbb{R}$ as a trigonometric polynomial (for more details, see (6.3) or (6.4)), with only 4 nonzero coefficients (see the 1st proof of Lemma 6.1 below); a typical sample of the corresponding nodal pictures are shown in Figure 2 (cf. § 6.2.) We may deduce that a.s. $\mathcal{N}(f_{\nu_0};\cdot)\equiv 0$, i.e. there are no compact domains of f_{ν_0} at all and all the domains are either predominantly horizontal or predominantly vertical, occurring with probability $\frac{1}{2}$. The analogous situation on the torus occurs for $n=m^2$ with

$$g_{0;m}(x)=\frac{1}{\sqrt{2}}\cdot\left(a_1\cdot\cos(m\cdot x_1+\eta_1)+a_2\cdot\cos(m\cdot x_2+\eta_2)\right),$$

where a_1,a_2 are Rayleigh(1) distributed independent random variables, and $\eta_1,\eta_2\in[0,2\pi)$ are random phases uniformly drawn in $[0,2\pi)$; in this case the nodal components in Figure 2 all become periodic with nontrivial homology, and their number is of order of magnitude

$$\mathcal{N}(g_0)\approx m\approx\sqrt{n}. \tag{2.2}$$

Since the Nazarov-Sodin constant does not vanish outside of the (tilted) Cilleruelo measure, for every finite instance f_n with $n\in S$ one would expect for more domains as compared to (2.2), whether or not

n is a square, i.e., $\mathcal{N}(f_n) \gg \sqrt{n}$. The above intuition has some reservations. A fragment of a sample nodal portrait of f_n with n corresponding to a measure μ_n close to Cilleruelo is given in Figure 1.

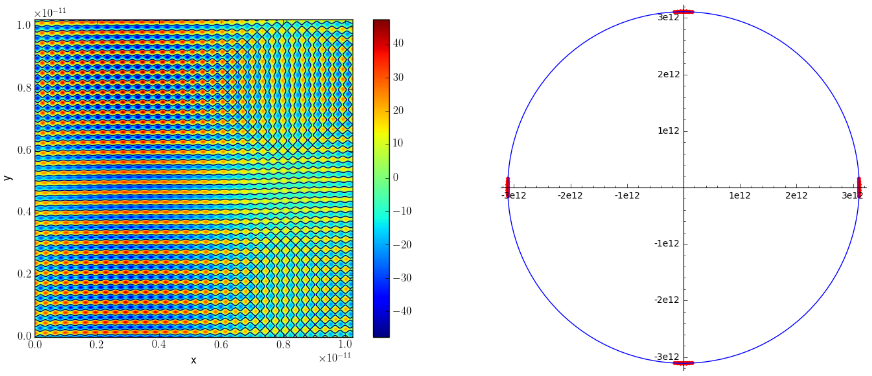

FIGURE 1. Left: Plot of a fragment of a random "Cilleruelo" type eigenfunction, nodal curves in black. Right: corresponding spectral measure. Here $n = 9676418088513347624474653$ and $r_2(n) = |\{(x, y) \in \mathbb{Z}^2 : x^2 + y^2 = n\}| = 256$; for this particular choice of n, the corresponding lattice points, shown in red, are concentrated around 4 antipodal points.

It exhibits that, just as in Figure 2, the nodal domains are all predominantly horizontal or vertical, but the suggested effect of the perturbed Cilleruelo shows that the periodic trajectories sometime connect in some percolation-like process, and transform from horizontal to vertical and back. Judging from the small presented fragment it seems difficult to determine to what extent this procedure decreases the total number of nodal domains, in particular whether the expectation is bounded or not. For a higher resolution picture, as well as some further examples of Cilleruelo eigenfunctions, see Appendix A.

In any case it is likely that the genuine asymptotic behavior of $\mathbb{E}[\mathcal{N}(f_{n_j})]$ depends on the rate of convergence (2.1), hence does not admit a simple asymptotic law. With all our reservations, the above discussion is our basis for the following question.

Question 2.2. Is it true that for an arbitrary Cilleruelo sequence,

$$\liminf_{j\to\infty} \mathbb{E}[N(f_{n_j})] \to \infty,$$

or, even stronger

$$\mathbb{E}[N(f_{n_j})] \gg \sqrt{n_j}?$$

If, as we tend to think, the answer to the latter question is "yes", then a simple compactness argument yields that for the full sequence $n \in S$ we have

$$\lim_{n\to\infty} \mathbb{E}[\mathcal{N}(f_n)] = \infty.$$

2.4. **The true nature of the Nazarov-Sodin constant.** Motivated by the fact that various local quantities, such as the nodal length variance [KKW], or the expected number of "flips" (see (3.6)) or critical points, only depends on the first non-trivial Fourier coefficient of the measure, we raise the following question.

Question 2.3. Is it true that $c_{NS}(\mu)$ with $\mu \in \mathcal{P}_{\text{symm}}$ only depends on finitely many Fourier coefficients, e.g. $\widehat{\mu}(4)$ or $(\widehat{\mu}(4), \widehat{\mu}(8))$?

2.5. **Key ideas of the proof of Theorem 1.3.** To prove the continuity of c_{NS} in Theorem 1.3 we wish to show that $|c_{NS}(\rho)-c_{NS}(\rho')|$ is small for two "close" spectral measures ρ, ρ'. To this end we show that for a large R there exists a coupling between the random fields f_ρ and $f_{\rho'}$, and that $\mathcal{N}(f_\rho, R)$ and $\mathcal{N}(f_{\rho'}, R)$ are very likely to be close (in fact, that f_ρ and $f_{\rho'}$ are C^1-close and that they have essentially the same nodal components). Of key importance is that both f_ρ and $f_{\rho'}$ are not only C^1-close, but also likely to be "stable" in the sense that small perturbations do not change the number of nodal components, except near the boundary. However, we can only prove stability, and the desired properties of the coupling, for square domains $\mathcal{D}_R$ for R *fixed*, and it is thus essential to have bounds on the difference

$$\mathcal{N}(f_\rho, R)/(4R^2) - c_{NS}(\rho)$$

that are *uniform* in both ρ and R.

To obtain uniformity in R we tile a "huge" square with a fixed "large" square, and count nodal domains entirely contained in the fixed large square. By translation invariance, the expectation over all large squares is the same, hence the scaled number of components in the large square (i.e., scaling by dividing by the area of the square) is the same as the scaled number of components of the huge square, up to an error involving the (scaled) number of nodal components that intersect a boundaries of at least one large square. This in turn can be *uniformly bounded* (in terms of ρ) by using Kac-Rice type techniques to uniformly bound the expected number of zeros of f_ρ lying on a curve (the bound of course depends on its length), cf. Lemma 3.2.

In case the huge square cannot exactly be tiled by large squares, we make use of the following observation: the number of nodal domains entirely contained in some region is bounded from above by the number of "flip points", i.e., points $x = (x_1, x_2)$ where $f = \frac{\partial f}{\partial x_1} = 0$, and the expected number of such points is, up to a uniform constant, bounded by the area of the region. To show this we again use a Kac-Rice type "local" estimates, cf. Lemma 3.4 and its proof.

Nazarov and Sodin assume that the support of ρ is not contained in a line, in order for non-degeneracy of $(f_\rho, \nabla f_\rho)$ to hold. Now, if $\rho_i \Rightarrow \rho$ and the limiting measure ρ is non-degenerate, there exists $\epsilon > 0$ such that ρ is outside a small neighborhood $\mathcal{P}_\epsilon$ of the degenerate measures within $\mathcal{P}$ defined in § 3 below (see 3.5). The outlined approach above yields continuity of c_{NS} around ρ in the complement $\mathcal{P} \setminus \mathcal{P}_\epsilon$.

On the other hand, if the limit ρ is degenerate we use a separate argument. First we show that $c_{NS}(\rho) = 0$ by showing that f_ρ, almost surely, has no bounded nodal domains; similarly this shows that we may assume that all ρ_i gives rise to non-degenerate fields. As non-degeneracy holds along the full sequence, we can then use Kac-Rice type local argument giving that $\mathbb{E}(\mathcal{N}(f_{\rho_i}; R))/R^2 \to 0$ as $i \to \infty$.

3. KAC-RICE PREMISE

We begin with collecting some notational conventions. Given a smooth function f on $\mathbb{R}^2$ let $f_1 = \partial_1 f = \frac{\partial f}{\partial x_1}$, $f_2 = \frac{\partial f}{\partial x_2}$, and $f_{12} = \partial_1 \partial_2 f = \frac{\partial^2 f}{\partial x_1 \partial x_2}$ (etc), where $x = (x_1, x_2) \in \mathbb{R}^2$; and similarly for smooth functions $f : \mathbb{T}^2 \to \mathbb{R}$.

The Kac-Rice formula is a standard tool for computing moments of various *local* properties of random (Gaussian) fields, such as, for example, number of nodal intersections against a reference curve, number of critical points etc. For our purposes we will not require any result beyond the expectation of the number of zeros $\mathcal{Z}_F$ of a stationary Gaussian field $F : \mathcal{D} \to \mathbb{R}^n$ on a compact domain $\mathcal{D} \subseteq \mathbb{R}^n$ (closed interval for $n = 1$), with the sole intention of applying it in the $2d$ case. For $x \in \mathcal{D}$ define the zero density as the conditional Gaussian expectation

$$K_1(x) = K_{1;F}(x) = \phi_{F(x)}(0) \cdot \mathbb{E}[|\det J_F(x)| \big| F(x) = 0], \tag{3.1}$$

where $\phi_{F(x)}$ is the probability density function of the Gaussian vector $F(x) \in \mathbb{R}^n$, and $J_F(x)$ is the Jacobian matrix of F at x. The Kac-Rice meta-theorem states that, under some non-degeneracy conditions on F,

$$\mathbb{E}[\mathcal{Z}_F] = \int\limits_{\mathcal{D}} K_1(x)dx; \tag{3.2}$$

to our best knowledge the mildest sufficient conditions for the validity of (3.2), due to Azais-Wschebor [AW, Theorem 6.3], is that for all $x \in \mathcal{D}$ the distribution of the Gaussian vector $F(x) \in \mathbb{R}^n$ is non-degenerate.

As for the zero density $K_1(x)$ in (3.1), since (3.1) is a Gaussian expectation depending on the law of $(F(x), J_F(x))$, it is in principle possible to express K_1 in terms of the covariance of F. For F a derived field of f_ρ (if, for example, F is a restriction of f_ρ on the reference curve $\mathcal{C}$ in Lemma 3.2 below, or $F = (f_\rho, \partial_1 f_\rho)$ in Lemma 3.4 below) it is possible to express $K_1(x)$ in terms of the covariance function

$$r_\rho(x,y) = \mathbb{E}[f_\rho(x) \cdot f_\rho(y)]$$

and its various derivatives, or, what is equivalent, its spectral measure ρ, supported on $B(1)$:

$$r_\rho(x) = \int\limits_{B(1)} e(\langle x, y\rangle) d\rho(y), \tag{3.3}$$

where

$$e(t) = e^{2\pi i t}.$$

In case F is stationary (see Lemma 3.4 below), $K_{1;F}(x)$ in (3.1) is independent of x; in this case to prove a uniform upper bound we only need to control it in terms of F.

For the Kac-Rice method to apply it is essential that the field is nondegenerate. In order to analyze certain degenerate limit measures we introduce the following notation. Given a stationary Gaussian field f_ρ with spectral measure ρ, let $C(\rho)$ denote the positive semi-definite covariance matrix

$$C(\rho) := \begin{pmatrix} \mathrm{Var}(\partial_1 f_\rho(0)) & \mathrm{Cov}(\partial_1 f_\rho(0), \partial_2 f_\rho(0)) \\ \mathrm{Cov}(\partial_1 f_\rho(0), \partial_2 f_\rho(0)) & \mathrm{Var}(\partial_2 f_\rho(0)) \end{pmatrix}, \tag{3.4}$$

and let $\lambda(\rho) \geq 0$ denote the smallest eigenvalue of $C(\rho)$. As the map $\rho \to C(\rho)$ is continuous, the same holds for $\rho \to \lambda(\rho)$. Thus, if we are given $\epsilon > 0$ define

$$\mathcal{P}_\epsilon := \{\rho \in \mathcal{P} : \lambda(\rho) < \epsilon\} \tag{3.5}$$

we find that $\mathcal{P} \setminus \mathcal{P}_\epsilon$ is a closed subset of $\mathcal{P}$. Abusing notation slightly it is convenient to let

$$\mathcal{P}_0 := \{\rho \in \mathcal{P} : \lambda(\rho) = 0\}$$

denote the set spectral measures giving rise to degenerate fields. (We may interpret the covariance matrix $C(\rho)$ as a matrix representing a positive semi-definite quadratic form; non-degeneracy is then equivalent to the form being positive definite. As quadratic forms in two variables can be diagonalised by a rotation, degeneracy implies that after a change of coordinates by rotation, we have $\mathrm{Var}(\partial_1 f) = \int \xi_1^2 \, d\rho(\xi) = 0$, and hence the support of ρ must be contained in the line $\xi_1 = 0$.)

As we intend to apply the Kac-Rice formula on f_ρ, for $\rho \in \mathcal{P} \setminus \mathcal{P}_\epsilon$ for some $\epsilon > 0$, and some derived random fields (see lemmas 3.2 and 3.4 below) we will need to collect the following facts.

Lemma 3.1. *(1) For every unit variance random field $F : \mathcal{D} \to \mathbb{R}$, and $x \in \mathcal{D}$, the value $F(x)$ is independent of the gradient $\nabla F(x)$.*

(2) The variances $\mathrm{Var}(\partial_1 f_\rho), \mathrm{Var}(\partial_2 f_\rho)$ of the first partial derivatives is bounded away from 0, uniformly for $\rho \in \mathcal{P} \setminus \mathcal{P}_\epsilon$.

The proof of Lemma 3.1 will be given in § 3.4.

3.1. **Intersections with curves and flips.** We begin with a bound on expected number of nodal intersections with curves, whose proof will be given in § 3.4.

Lemma 3.2. *Let $\mathcal{C} \subseteq \mathbb{R}^2$ be a smooth curve of length $\mathcal{L}$, and $\mathcal{N}(f_\rho, \mathcal{C})$ the number of nodal intersections of f_ρ with $\mathcal{C}$, $\rho \in \mathcal{P}$. Then*

$$\mathbb{E}[\mathcal{N}(f_\rho, \mathcal{C})] = O(\mathcal{L})$$

with constant involved in the 'O'-notation absolute.

The notion of "nodal flips" will be very useful for giving uniform upper bounds on the number of nodal domains.

Notation 3.3. For $\mathcal{D} \subseteq \mathbb{R}^2$ a nice closed domain we denote the number of vertical and horizontal nodal flips

$$\begin{aligned} S_1(f_\rho; \mathcal{D}) &= \#\{x \in \mathcal{D} : \ f_\rho(x) = \partial_1 f_\rho(x) = 0\}, \\ S_2(f_\rho; \mathcal{D}) &= \#\{x \in \mathcal{D} : \ f_\rho(x) = \partial_2 f_\rho(x) = 0\}, \end{aligned} \tag{3.6}$$

respectively.

Lemma 3.4. *For all $\rho \in \mathcal{P} \setminus \mathcal{P}_0$, we have*

$$\begin{aligned} \mathbb{E}[S_1(f_\rho; \mathcal{D})] &= O\left(\text{Area}(\mathcal{D}) \cdot \text{Var}(\partial_2 f_\rho)^{1/2}\right), \\ \mathbb{E}[S_2(f_\rho; \mathcal{D})] &= \left(\text{Area}(\mathcal{D}) \cdot \text{Var}(\partial_1 f_\rho)^{1/2}\right), \end{aligned} \tag{3.7}$$

and consequently

$$\max(\mathbb{E}[S_1(f_\rho; \mathcal{D})], \mathbb{E}[S_2(f_\rho; \mathcal{D})]) = O(\text{Area}(\mathcal{D}))$$

with constants involved in the 'O'-notation absolute.

Lemma 3.4 will be proved in § 3.4. As it was mentioned above, for $\rho \in \mathcal{P}_\epsilon$ we may arrange that, after rotating if necessary, either $\text{Var}(\partial_1 f_\rho)$ or $\text{Var}(\partial_2 f_\rho)$ is at most ϵ. To treat the degenerate case $\rho \in \mathcal{P}_0$ we record the following fact.

Lemma 3.5. *If $\rho \in \mathcal{P}_0$ then $\mathcal{N}(f_\rho; \cdot) \equiv 0$. In particular in this case* (1.4) *holds with $c_{NS}(\rho) = 0$.*

Proof. After changing coordinates by a rotation, we may assume that $\text{Var}(\partial_1 f) = 0$. Hence, almost surely, we have $f(x_1, x_2) = g(x_2)$ for some function g, and thus f has no compact nodal domains, and in particular $c_{NS}(\rho) = 0$. □

3.2. **Proof of Proposition 1.1.**

Proof. First, we may assume that $\rho \in \mathcal{P} \setminus \mathcal{P}_0$ by the virtue of Lemma 3.5, so that we are eligible to apply Lemma 3.4 on f_ρ. Now let $R_2 \ggg R_1 \ggg 0$ be a two big real numbers; for notational convenience we will at first assume that

$$R_2 = kR_1 \tag{3.8}$$

is an integer multiple of R_2, $k \gg 1$. We divide the square $\mathcal{D}_{R_2} = [-R_2, R_2]^2$ into $4k^2 = 4\frac{R_2^2}{R_1^2}$ smaller squares

$$\{\mathcal{D}_{R_2;i,j}\}_{i,j=1\dots 2k}$$

of side length R_1, disjoint save to boundary overlaps. Every nodal component lying in $\mathcal{D}_{R_2}$ is either lying entirely in one of the $\mathcal{D}_{R_2;i,j}$ or intersects at least one of the vertical or horizontal line segments, $\{x = i \cdot T, \, |y| \le R_2\}$, $i = -k, \dots k$, or $\{y = j \cdot T, \, |x| \le R_2\}$, $j = -k, \dots, k$ respectively. Let $\mathcal{N}(f_\rho; \mathcal{D}_{R_2;i,j})$ be the number of nodal components of f_ρ lying in $\mathcal{D}_{R_2;i,j}$, and $\mathcal{Z}(f_\rho; R_2, x = iR_1)$, $\mathcal{Z}(f_\rho; R_2, y = jR_1)$ be the number of nodal intersections of f_ρ against a finite vertical or horizontal line segment as above.

The above approach shows that

$$\mathcal{N}(f_\rho; R_2) = \sum_{1\le i,j\le 2k} \mathcal{N}(f_\rho; \mathcal{D}_{R_2;i,j}) + O\left(\sum_{i=-k}^{k} \mathcal{Z}(f_\rho; R_2, x = iR_1)\right) + O\left(\sum_{j=-k}^{k} \mathcal{Z}(f_\rho; R_2, y = jR_1)\right). \tag{3.9}$$

We now take expectation of both sides of (3.9); using the translation invariance of f_ρ, and Lemma 3.2 we find that

$$\mathbb{E}[\mathcal{N}(f_\rho; R_2)] = 4k^2 \cdot \mathbb{E}[\mathcal{N}(f_\rho; R_1/2)] + O(kR_2), \tag{3.10}$$

where the constant involved in the 'O'-notation is absolute. A simple manipulation with (3.10), bearing in mind (3.8), now implies

$$\left|\frac{\mathbb{E}[\mathcal{N}(f_\rho; R_2)]}{4R_2^2} - \frac{\mathbb{E}[\mathcal{N}(f_\rho; R_1/2)]}{R_1^2}\right| = O\left(\frac{1}{R_1}\right) \tag{3.11}$$

with the constant involved in the 'O'-notation absolute, with (3.11).

In case R_2 is not an integer multiple of R_1, in the above argument we leave a small rectangular corridor of size at most $R_1 \times R_2$ (in fact, two such corridors). In this case the estimate (3.11) should be replaced by

$$\left|\frac{\mathbb{E}[\mathcal{N}(f_\rho; R_2)]}{4R_2^2} - \frac{\mathbb{E}[\mathcal{N}(f_\rho; R_1/2)]}{R_1^2}\right| = O\left(\frac{1}{R_1} + \frac{R_1}{R_2}\right), \tag{3.12}$$

with $O(\frac{R_1}{R_2})$ coming from the contribution of the small rectangular leftover corridor thinking of R_2 much bigger than R_1; here we used Lemma 3.4, valid since we assumed $\rho \in \mathcal{P} \setminus \mathcal{P}_0$. The latter estimate (3.12) shows that $\left\{\frac{\mathbb{E}[\mathcal{N}(f_\rho;R)]}{4R^2}\right\}$ satisfies the Cauchy convergence criterion (if R_1 and R_2 are of comparable size then we use the triangle inequality, after tiling both $\mathcal{D}_{R_2}$ and $\mathcal{D}_{R_1/2}$ with much finer mesh size), we then denote its limit by

$$c_{NS}(\rho) := \lim_{R\to\infty} \frac{\mathbb{E}[\mathcal{N}(f_\rho; R)]}{4R^2}.$$

Now that the existence of the limit $c_{NS}(\rho)$ is proved, we may assume that R_2 is an integer multiple of R_1, and take the limit $R_2 \to \infty$ in (3.11); it yields

$$\left|\frac{\mathbb{E}[\mathcal{N}(f_\rho; R_1/2)]}{R_1^2} - c_{NS}(\rho)\right| = O\left(\frac{1}{R_1}\right), \tag{3.13}$$

again with the constant in the'O'-notation on the r.h.s. absolute. We conclude the proof of Proposition 1.1 by noticing that (3.13) is a restatement of (1.4) (e.g. replace R_1 by $R/2$).

□

3.3. **Proof of Proposition 1.2.**

Proof. Again, for $\rho \in \mathcal{P}_0$ there is nothing to prove here thanks to Lemma 3.5, so that from this point on we assume that $\rho \in \mathcal{P} \setminus \mathcal{P}_0$. Let

$$\xi = \xi(\rho) := \liminf_{R\to\infty} \mathbb{E}\left[\left|\frac{\mathcal{N}(f_\rho; R)}{4R^2} - c_{NS}(\rho)\right|\right]; \tag{3.14}$$

in what follows we argue that, in fact, ξ is the limit. Let $\epsilon > 0$ be given and $R_1 = R_1(\rho, \epsilon) > 0$ such that

$$\mathbb{E}\left[\left|\frac{\mathcal{N}(f_\rho; R_1/2)}{R_1^2} - c_{NS}(\rho)\right|\right] < \xi + \epsilon. \tag{3.15}$$

Following the proof of Proposition 1.1 let $R_2 \gg R_1 \gg 0$ be a large real number; as before we divide the square $\mathcal{D}_{R_2} = [-R_2, R_2]^2$ into the smaller squares $\{\mathcal{D}_{R_2,i,j}\}_{1\le i,j\le 2k}$ of side length R_1 leaving a couple of corridors of size at most $R_1 \times R_2$, and write (cf. 3.9)

$$\begin{aligned} 0 \le \mathcal{N}(f_\rho; R_2) - \sum_{1\le i,j\le 2k} \mathcal{N}(f_\rho; \mathcal{D}_{R_2;i,j}) &\le \sum_{i=-k}^{k} \mathcal{Z}(f_\rho; R_2, x = iR_1) \\ + \sum_{j=-k}^{k} \mathcal{Z}(f_\rho; R_2, y = jR_1) + \mathcal{N}(f_\rho; \mathcal{F}_{R_2,R_1}), \end{aligned} \tag{3.16}$$

where we denoted $\mathcal{F}_{R_2,R_1}$ to be the union of the two leftover rectangular corridors, and $\mathcal{N}(f_\rho, \mathcal{F}_{R_2,R_1})$ the corresponding number of nodal domains lying entirely inside $\mathcal{F}_{R_2,R_1}$.

Taking the expectation of both sides of (3.16) and dividing by $4R_2^2$ we have that (using the non-negativity of the l.h.s. of (3.16))

$$\mathbb{E}\left[\left|\frac{\mathcal{N}(f_\rho; R_2)}{4R_2^2} - \frac{1}{4k^2} \cdot \sum_{1\le i,j\le 2k} \frac{\mathcal{N}(f_\rho; \mathcal{D}_{R_2;i,j})}{R_1^2}\right|\right] = O\left(\frac{1}{R_1} + \frac{R_1}{R_2}\right), \tag{3.17}$$

thanks to Lemma 3.4, valid for $\rho \in \mathcal{P} \setminus \mathcal{P}_0$. On the other hand, by (3.15), the triangle inequality, and the translation invariance of f_ρ, we have that

$$\begin{aligned} &\mathbb{E}\left[\left|\frac{1}{4k^2} \cdot \sum_{1\le i,j\le 2k} \frac{\mathcal{N}(f_\rho; \mathcal{D}_{R_2;i,j})}{R_1^2} - c_{NS}(\rho)\right|\right] \\ &\le \frac{1}{4k^2} \sum_{1\le i,j\le 2k} \mathbb{E}\left[\left|\frac{\mathcal{N}(f_\rho; \mathcal{D}_{R_2;i,j})}{R_1^2} - c_{NS}(\rho)\right|\right] \\ &= \mathbb{E}\left[\left|\frac{\mathcal{N}(f_\rho; R_1/2)}{R_1^2} - c_{NS}(\rho)\right|\right] < \xi + \epsilon. \end{aligned} \tag{3.18}$$

We have then

$$\begin{aligned} \mathbb{E}\left[\left|\frac{\mathcal{N}(f_\rho; R_2)}{4R_2^2} - c_{NS}(\rho)\right|\right] &\le \mathbb{E}\left[\left|\frac{\mathcal{N}(f_\rho; R_2)}{4R_2^2} - \frac{1}{4k^2} \cdot \sum_{1\le i,j\le 2k} \frac{\mathcal{N}(f_\rho; \mathcal{D}_{R_2;i,j})}{R_1^2}\right|\right] \\ + \mathbb{E}\left[\left|\frac{1}{4k^2} \cdot \sum_{1\le i,j\le 2k} \frac{\mathcal{N}(f_\rho; \mathcal{D}_{R_2;i,j})}{R_1^2} - c_{NS}(\rho)\right|\right] &< \xi + \epsilon + O\left(\frac{1}{R_1} + \frac{R_1}{R_2}\right). \end{aligned} \tag{3.19}$$

by Lemma 3.4, (3.17), (3.18), and, again, the triangle inequality. Since $\epsilon > 0$ on the r.h.s. of (3.19) is arbitrary, fixing $R_1 \gg 0$ satisfying (3.15) arbitrarily big, and taking $\limsup\limits_{R_2\to\infty}$ of both sides of (3.19) yields

$$\limsup_{R\to\infty} \mathbb{E}\left[\left|\frac{\mathcal{N}(f_\rho; R)}{4R^2} - c_{NS}(\rho)\right|\right] \le \xi;$$

comparing the latter equality with (3.14) finally yields the existence of the limit (1.5).

□

3.4. **Proofs of the local estimates.**

Proof of Lemma 3.1. Let $r_F(x,y) = \mathbb{E}[F(x)\cdot F(y)]$ be the covariance function of F, by the assumptions of Lemma 3.1 we have that

$$\mathbb{E}[f(x)\cdot f(x)] = r_F(x,x) \equiv 1. \tag{3.20}$$

The independence of $F(x)$ and $\nabla F(x)$ then follows upon differentiating (3.20) concluding the first part of Lemma 3.1. The second part of Lemma 3.1 is obvious from the definition (3.4) of $C(\rho)$ bearing in mind the aforementioned diagonalisation of $C(\rho)$ by a rotation (see the interpretation of $C(\rho)$ and $\mathcal{P}_\epsilon$ immediately after (3.5)).

□

Proof of Lemma 3.2. Let $\gamma : [0,\mathcal{L}] \to \mathbb{R}^2$ be an arc-length parametrization of $\mathcal{C}$, and

$$g(t) = g_{\mathcal{C},\rho}(t) = f_\rho(\gamma(t))$$

be the restriction $g : [0,\mathcal{L}] \to \mathbb{R}$ of f_ρ along $\mathcal{C}$. The process g is centred Gaussian, with covariance function

$$r_g(t_1,t_2) = r_\rho(\gamma(t_2) - \gamma(t_1)) \tag{3.21}$$

with r_ρ the covariance function of f_ρ.

The number of nodal intersections of f_ρ against $\mathcal{C}$ is then a.s. equal to $\mathcal{N}(f_\rho,\mathcal{C}) = \mathcal{Z}_g$, the number of zeros of g on $[0,\mathcal{L}]$. Since f_ρ has unit variance, so does g; therefore (Lemma 3.1) for every $t\in[0,\mathcal{L}]$ the value $g(t)$ is independent of the derivative $g'(t)$. We then have by Kac-Rice [AW, Theorem 6.3]

$$\mathbb{E}[\mathbb{Z}_g] = \int_0^{\mathcal{L}} K_1(t)dt,$$

where

$$K_1(t) = K_{1;g}(t) = \frac{1}{\pi}\sqrt{\left.\frac{\partial^2 r_g}{\partial t_1 \partial t_2}\right|_{t_1=t_2=t}}$$

is the zero density of g. The statement of Lemma 3.2 will follow once we show that the mixed second derivative $\frac{\partial^2 r_g}{\partial t_1\partial t_2}$ of r_g is uniformly bounded by an absolute constant, independent of γ and $\rho\in\mathcal{P}$.

To this end we differentiate (3.21) to compute

$$\partial_{t_1}\partial_{t_2} r_g(t_1,t_2) = -\dot\gamma(t_1)\cdot H_{r_\rho}(\gamma(t_2)-\gamma(t_1))\cdot\dot\gamma(t_2)^t,$$

where H_{r_ρ} is the Hessian of r_ρ. That

$$\partial_{t_1}\partial_{t_2} r_g(t_1,t_2)$$

is bounded by an absolute constant then follows from the fact that

$$\|\dot\gamma(t_1)\| = \|\dot\gamma(t_2)\| = 1,$$

and that H_{r_ρ} is bounded follows by differentiating (3.3), using the bounded support of ρ.

□

Proof of Lemma 3.4. To prove the first assertion we record the following useful fact about nondegenerate centred Gaussians: with (X,Y,Z) denoting components of a nondegenerate multivariate normal distribution having mean zero, we have

$$\mathrm{Var}(X|Y=Z=0) \le \mathrm{Var}(X). \tag{3.22}$$

While it is easy to validate (3.22) by an explicit computation, it is also a (very) particular consequence of the vastly general Gaussian Correlation Inequality [Roy].

Now, by Kac-Rice [AW, Theorem 6.3] it follows that, if for all $x \in \mathcal{D}$, the Gaussian distribution of

$$F(x) := (f_\rho(x), \partial_1 f_\rho(x)) \tag{3.23}$$

is non-degenerate (holding by both parts of Lemma 3.1), then (3.2) is satisfied with

$$K_1(x) = K_{1;\rho}(x)$$

the appropriately defined flips density (3.1) with F given by (3.23), and by stationarity we have

$$K_1(x) \equiv K_1(0). \tag{3.24}$$

Now from (3.24) and (3.2) it then follows that

$$\mathbb{E}[S_1(f_\rho; \mathcal{D})] = K_1(0) \cdot \mathrm{Area}(\mathcal{D}), \tag{3.25}$$

and it is sufficient to show that

$$K_1(0) = O\left(\mathrm{Var}(\partial_2 f_\rho)^{1/2}\right).$$

Upon recalling that F is given by (3.23) we have that

$$K_1(0) = \phi_{F(0)}(0,0) \cdot \mathbb{E}[|\det J_F(0)| \big| f_\rho(0) = \partial_1 f_\rho(0) = 0], \tag{3.26}$$

where $\phi_{F(0)}$ is the probability density of the Gaussian vector

$$F(0) = (f_\rho(0), \partial_1 f_\rho(0)),$$

and

$$J_F(x) = \begin{pmatrix} \partial_1 f_\rho(x) & \partial_2 f_\rho(x) \\ \partial_1^2 f_\rho(x) & \partial_1 \partial_2 f_\rho(x) \end{pmatrix}$$

is the Jacobian matrix of F.

Conditioned on $\partial_1 f_\rho(0) = 0$ we have that

$$\det(J_F(x)) = -\partial_2 f_\rho(x) \cdot \partial_1^2 f_\rho(x),$$

hence (3.26) is

$$\begin{aligned} K_1(0) &= \frac{1}{2\pi\sqrt{\mathrm{Var}(\partial_1 f_\rho(0))}} \cdot \mathbb{E}[|\partial_2 f_\rho(0) \cdot \partial_1^2 f_\rho(0)| \big| f_\rho(0) = \partial_1 f_\rho(0) = 0] \\ &\le \frac{1}{2\pi\sqrt{\mathrm{Var}(\partial_1 f_\rho(0))}} \cdot \sqrt{\mathrm{Var}(\partial_2 f_\rho(0) \big| f_\rho(0) = \partial_1 f_\rho(0) = 0)} \times \\ &\times \sqrt{\mathrm{Var}(\partial_1^2 f_\rho(0) \big| f_\rho(0) = \partial_1 f_\rho(0) = 0)} \\ &= O\left(\frac{\sqrt{\mathrm{Var}(\partial_2 f_\rho(0))} \times \sqrt{\mathrm{Var}(\partial_1^2 f_\rho(0))}}{\sqrt{\mathrm{Var}(\partial_1 f_\rho(0))}} \right) \end{aligned} \tag{3.27}$$

by Cauchy-Schwartz and the above mentioned bound (3.22) on the conditional variance.

Now, differentiating (3.3) we have that

$$\mathrm{Var}(\partial_1 f_\rho(0)) = (2\pi)^2 \int_{B(1)} y_1^2 d\rho(y) \text{ and } \mathrm{Var}(\partial_2 f_\rho(0)) = (2\pi)^2 \int_{B(1)} y_2^2 d\rho(y), \tag{3.28}$$

showing in particular the uniform bound

$$\mathrm{Var}(\partial_2 f_\rho(0)) = O(1). \tag{3.29}$$

Differentiating (3.3) in a similar fashion we obtain the analogous expression

$$\mathrm{Var}(\partial_1^2 f_\rho(0)) = (2\pi)^4 \int_{B(1)} y_1^4 d\rho(y), \tag{3.30}$$

for $\mathrm{Var}(\partial_1^2 f_\rho(0))$. The identity (3.30) together with (3.28) imply that the ratio

$$\frac{\mathrm{Var}(\partial_1^2 f_\rho(0))}{\mathrm{Var}(\partial_1 f_\rho(0))} = O(1) \tag{3.31}$$

is uniformly bounded, since $y_1^4 \le y_1^2$ for all $y \in B(1)$. Finally (3.31) together with (3.29) imply that the r.h.s. of (3.27) is uniformly bounded, sufficient for the first assertion of Lemma 3.4 via (3.25).

The second assertion of Lemma 3.4 can be deduced from the first by changing coordinates via rotating by $\pi/2$. The final assertion follows immediately from the two first.

□

4. Proof of Theorem 1.3: continuity of the Nazarov-Sodin constant

We shall treat the case of limiting spectral measures ρ lying in $\mathcal{P}_0$ separately, and we begin with the following result.

Lemma 4.1. *If $\rho \in \mathcal{P}_0$ and $\rho_i \Rightarrow \rho$ (convergence in weak-* topology), then*

$$c_{NS}(\rho_i) \to c_{NS}(\rho) = 0.$$

Proof. By Lemma 3.5 we have $c_{NS}(\rho) = 0$. Moreover, the same holds for those j such that $\rho_j \in \mathcal{P}_0$ and hence it is enough to treat the case that $\rho_j \notin \mathcal{P}_0$ for all j. Now, as $\rho_j \to \rho$ and $\rho \in \mathcal{P}_0$, we find that given $\epsilon > 0$ there exist J such that $\rho_j \in \mathcal{P}_\epsilon$ for all $j \ge J$. Thus, after making a (possibly j-dependent) rotational change of variables, we may assume that $\mathrm{Var}(\partial_1 f_{\rho_j}) \le \epsilon$ and Lemma 3.4 then implies that $c_{NS}(\rho_j) = O(\sqrt{\epsilon})$ for $j \ge J$. The result follows.

□

4.1. **Preliminary results.**

4.1.1. *Perturbing the random field.* The following proposition, proved in § 5 below, will be used in the proof of Theorem 1.3.

Proposition 4.2. *Let $R > 0$ be sufficiently big, $\epsilon > 0$, $\xi > 0$, and let $\{\rho_j\} \subseteq \mathcal{P} \setminus \mathcal{P}_\epsilon$ be a sequence of probability measures, weak-$*$ convergent to $\rho_0 \in \mathcal{P} \setminus \mathcal{P}_\epsilon$. There exists a number $j_0 = j_0(\epsilon; \{\rho_j\}; R, \xi) > 0$ such that for all $j > j_0$ there exists a coupling of f_{ρ_j} and f_{ρ_0} and an event $\Omega_0 = \Omega_0(\rho_0, \rho_j; R, \xi)$ of probability $\mathcal{P}(\Omega_0) < \xi$ such that on $\Omega \setminus \Omega_0$ we have*

$$\mathcal{N}(f_{\rho_j}; R-1) \le \mathcal{N}(f_{\rho_0}; R) \le \mathcal{N}(f_{\rho_j}; R+1). \tag{4.1}$$

4.1.2. *Small domains.* For smooth (deterministic) function $F : \mathbb{R}^2 \to \mathbb{R}$, $R > 0$ and a small parameter $\delta \in (0, 1]$ we denote $\mathcal{N}_{\delta-sm}(F; R)$ to be the number of domains of area $< \delta$ ("δ-small domains") lying entirely inside $B(R)$. Accordingly, let

$$\mathcal{N}_{\delta-big}(F; R) = \mathcal{N}(F; R) - \mathcal{N}_{\delta-sm}(F; R)$$

be the number of "δ-big domains" (a more appropriate, though cumbersome, term would be "δ-not-small"). We have the following bound for the expected number of δ-small domains of f_ρ.

Lemma 4.3 (Cf. [NS2]; [SW] Lemma 4.12)**.** *For every $\epsilon > 0$ there exist constants $c_0(\epsilon), C_0(\epsilon) > 0$ such that the expected number of δ-small domains satisfies*

$$\sup_{R \ge 10} \frac{\mathbb{E}[\mathcal{N}_{\delta-sm}(f_\rho; R)]}{R^2} \le C_0(\epsilon) \cdot \delta^{c_0(\epsilon)}$$

uniformly for all $\rho \in \mathcal{P} \setminus \mathcal{P}_\epsilon$.

Proof. If $\rho \in \mathcal{P} \setminus \mathcal{P}_0$ then the non-degeneracy conditions of [NS2, Lemma 9] (or [SW, Proposition 4.12]) are satisfied, hence these imply that in this case there exist constants $C_0(\rho)$ and $c_0(\rho)$, depending continuously on $\rho \in \mathcal{P} \setminus \mathcal{P}_0$, so that

$$\sup_{R\geq 10} \frac{\mathbb{E}[\mathcal{N}_{\delta-sm}(f_\rho; R)]}{R^2} \leq C_0(\rho) \cdot \delta^{c_0(\rho)}.$$

The uniformity for choice of $C(\epsilon), c(\epsilon)$ then follows from the compactness of $\mathcal{P} \setminus \mathcal{P}_\epsilon \subseteq \mathcal{P} \setminus \mathcal{P}_0$.

□

4.2. **Proof of Theorem 1.3.**

Proof of Theorem 1.3. Let $\{\rho_j\}_{j\geq 1} \subseteq \mathcal{P}$ be a sequence of probability measures weak-$*$ converging to $\rho_0 \in \mathcal{P}$; the statement of Theorem 1.3 is that in this situation the corresponding Nazarov-Sodin constants

$$c_{NS}(\rho_j) \to c_{NS}(\rho_0) \tag{4.2}$$

converge to the Nazarov-Sodin constant of ρ_0.

The case of $\rho_0 \in \mathcal{P}_0$ follows from Lemma 4.1. For $\rho_0 \notin \mathcal{P}_0$, we have $\rho_0 \in \mathcal{P} \setminus \mathcal{P}_{2\epsilon}$ for some $\epsilon > 0$ and thus we may assume that $\rho_j \in \mathcal{P} \setminus \mathcal{P}_\epsilon$ for all sufficiently large j; without loss of generality we may assume that $\rho_j \in \mathcal{P} \setminus \mathcal{P}_\epsilon$ for all j.

Proposition 1.1 yields that given $\alpha > 0$ there exists

$$R_0 = R_0(\alpha) \gg 0$$

sufficiently big so that for all $R > R_0$ and all $\rho \in \mathcal{P}$ we have

$$\left| \frac{\mathbb{E}[\mathcal{N}(f_\rho, R)]}{4R^2} - c_{NS}(\rho) \right| < \alpha; \tag{4.3}$$

in particular (4.3) applies to $\rho = \rho_j$ with $j \geq 1$ or $\rho = \rho_0$. We now apply Proposition 4.2 with $R > R_0$, and $\xi > 0$ small, so that it yields a number j_0 sufficiently big such that for all $j > j_0$ there exists an event $\Omega_0 = \Omega_0(R, j, \xi)$ of probability

$$\mathcal{P}(\Omega_0) > 1 - \xi, \tag{4.4}$$

such that on Ω_0 we have

$$\mathcal{N}(f_{\rho_j}; R - 1) \leq \mathcal{N}(f_{\rho_0}; R) \leq \mathcal{N}(f_{\rho_j}; R + 1). \tag{4.5}$$

We are now going to show that the difference

$$\mathbb{E}[\mathcal{N}(f_{\rho_j}; R + 1)] - \mathbb{E}[\mathcal{N}(f_{\rho_j}; R - 1)] \geq 0$$

is small (compared to R^2) for $R \to \infty$; this would also imply that

$$\mathbb{E}[\mathcal{N}(f_{\rho_j}; R)] - \mathbb{E}[\mathcal{N}(f_{\rho_0}; R)]$$

is small (compared to R^2), and thus $c_{NS}(\rho_j) - c_{NS}(\rho_0)$ is small via (4.3). Recall that $\mathcal{D}_R$ is the square

$$\mathcal{D}_R := [-R, R]^2 \subseteq \mathbb{R}^2,$$

and denote

$$\mathcal{A}_R = \mathcal{D}_{R+1} \setminus \text{Int}(\mathcal{D}_{R-1})$$

to be the strip lying inside the $2(R+1)$-side square, outside the $2(R-1)$-side square. If for some $\rho \in \mathcal{P}$ a nodal domain of f_ρ is lying entirely inside $\mathcal{D}_{R+1}$ but not $\mathcal{D}_{R-1}$, then that nodal domain is either entirely lying inside $\mathcal{A}_R$ or intersects the boundary $\partial\mathcal{D}_{R-1}$ of the smaller of the squares. In either case that nodal domain necessarily contains either a horizontal or a vertical flip lying in $\mathcal{A}_R$, i.e. a point $x \in \mathcal{A}_R$ such that either $f_\rho(x) = f_{\rho;1}(x) = 0$ or $f_\rho(x) = f_{\rho;2}(x) = 0$, that is, recalling the notation (3.6) of nodal flips numbers, we have

$$0 \le \mathcal{N}(f_\rho; R+1) - \mathcal{N}(f_\rho; R-1) \le S_1(f_\rho; \mathcal{A}_R) + S_2(f_\rho; \mathcal{A}_R),$$

and upon taking the expectations of both sides of the latter inequality we obtain

$$0 \le \mathbb{E}[\mathcal{N}(f_\rho; R+1)] - \mathbb{E}[\mathcal{N}(f_\rho; R-1)] \le \mathbb{E}[S_1(f_\rho; \mathcal{A}_R) + S_2(f_\rho; \mathcal{A}_R)] \le C_1 \cdot R \tag{4.6}$$

by Lemma 3.4, with $C_1 > 0$ an absolute constant.

Now let $\delta > 0$ be a small parameter and recall the definition of δ-small and δ-big domains counts in § 4.1.2. We invoke (4.5) via (4.6), together with Lemma 4.3, and obtain that (for $j > j_0$)

$$\begin{aligned} &\mathbb{E}\left[\left|\mathcal{N}(f_{\rho_j}; R) - \mathcal{N}(f_{\rho_0}; R)\right|\right] \\ &\le C_2 \left(R + R^2\delta^{c_0} + \int\limits_{\Omega\setminus\Omega_0} \left(\mathcal{N}_{\delta-big}(f_{\rho_0}; 2R) + \mathcal{N}_{\delta-big}(f_{\rho_j}; 2R)\right) d\mathcal{P}(\omega) \right) \\ &\le C_2 \left(R + R^2\delta^{c_0} + \frac{8}{\delta} R^2 \cdot \mathcal{P}(\Omega \setminus \Omega_0) \right). \end{aligned}$$

for $C_2 > 0$ an absolute constant. Recalling (4.4) the above implies

$$\frac{\mathbb{E}\left[\left|\mathcal{N}(f_{\rho_j}; R) - \mathcal{N}(f_{\rho_0}; R)\right|\right]}{R^2} \le C_2 \left(\frac{1}{R} + \delta^{c_0} + \frac{8\xi}{\delta} \right).$$

Now using the triangle inequality with (4.3) applied on ρ_j and ρ_0 implies that for $j > j_0$ one has

$$|c_{NS}(\rho_j) - c_{NS}(\rho_0)| \le C_2 \left(\frac{1}{R} + \delta^{c_0} + \frac{8\xi}{\delta} + 2\alpha \right). \tag{4.7}$$

Since the r.h.s. (and thus the l.h.s.) of (4.7) can be made arbitrarily small by first choosing the parameters α and δ sufficiently small, and then $R > R_0(\alpha)$ sufficiently large, and finally ξ sufficiently small, and in light of the fact that the l.h.s. of (4.7) does not depend on R, this yields (4.2). As mentioned above, this is equivalent to the statement of Theorem 1.3.

□

Remark 4.4. The above argument can be simplified in the case of monochromatic waves, as here small domains do not exist by an application of the Faber-Krahn inequality [Ma, Theorem 1.5], so there is no need to invoke Lemma 4.3 to bound their contribution.

5. Proof of Proposition 4.2

The ultimate goal of this section is giving a proof for Proposition 4.2. Towards this goal we first construct the exceptional event Ω_0 in (5.4) below; it will consist of various sub-events defined in § 5.1 that would guarantee that on $\Omega \setminus \Omega_0$ both fields f_{ρ_0} and f_{ρ_j} (for j sufficiently big) are “stable” in the sense that a small perturbation of our function has a minor effect on its nodal structure, and that the perturbation $f_{\rho_0} - f_{\rho_j}$ is small in a sense to be made precise. That Ω_0 is *rare* is established in § 5.2. Proposition 4.2 will be finally proved in § 5.3 assuming some auxiliary results that will be established in § 5.4.

5.1. **Constructing the exceptional event Ω_0.**

Definition 5.1. (1) For $R > 0$ big parameter, $\beta > 0$ small parameter, and $\rho \in \mathcal{P}$ we define the "unstable" event

$$\Omega_1(f_\rho; R, \beta) := \left\{ \min_{x \in B(2R)} \max\{|f_\rho(x)|, \|\nabla f_\rho(x)\|\} \le 2\beta \right\},$$

i.e., that there exists a point in the ball $B(2R)$ such that both f_ρ and its gradient are small.

(2) For $R, M > 0$ big parameters, $\rho \in \mathcal{P}$ we define

$$\Omega_2(f_\rho; R, M) := \left\{ \|f_\rho\|_{C^2(B(2R))} \ge M \right\}.$$

(3) Let $\rho, \rho' \in \mathcal{P}$ be two measures and $f_\rho, f_{\rho'}$ copies of the corresponding random fields on $\mathbb{R}^2$ defined on the same probability space Ω. For $R > 0$, $\beta > 0$ define

$$\Omega_3(f_\rho, f_{\rho'}; R, \beta) := \left\{ \|f_\rho - f_{\rho'}\|_{C^1(B(2R))} \ge \beta \right\}.$$

5.2. **The exceptional event is rare.** We present the following auxiliary lemmas 5.2-5.4 which together imply that the exceptional event is rare. Lemmas 5.2-5.4 will be proved in § 5.4.

Lemma 5.2 (Cf. [So], Lemma 5)**.** *For every $\rho \in \mathcal{P} \setminus \mathcal{P}_\epsilon$, $R > 0$, $M > 0$ and $\xi > 0$ there exists a number $\beta = \beta(\epsilon; \rho; R, \xi) > 0$ such that the probability of $\Omega_1(f_\rho; R, \beta)$ outside of $\Omega_2(f_\rho; R, M)$ is*

$$\mathcal{P}(\Omega_1(f_\rho; R, \beta) \setminus \Omega_2(f_\rho; R, M)) < \xi.$$

Lemma 5.3. *(1) For every $\rho \in \mathcal{P}$, $R > 0$ and $\xi > 0$ there exists a number $M = M(f_\rho; R, \xi)$ so that*

$$\mathcal{P}(\Omega_2(f_\rho; R, M)) < \xi.$$

(2) Let $R > 0$ be sufficiently big, $\xi > 0$, and a sequence $\{\rho_j\} \subseteq \mathcal{P}$ of probability measures, weak-$*$ convergent to $\rho_0 \in \mathcal{P}$. Then there exists a number $M = M(\rho_0; R, \xi) > 0$ and $j_0 = j_0(f_{\rho_0}; R, \xi)$ such that for all $j > j_0$ we have

$$\mathcal{P}(\Omega_2(f_{\rho_j}; R, M)) < \xi.$$

Lemma 5.4 (Cf. [So], Lemma 4)**.** *Let $R > 0$ be sufficiently big, $M > 0$, $\beta > 0$, $\xi > 0$, and a sequence $\{\rho_j\} \subseteq \mathcal{P}$ of probability measures, weak-$*$ convergent to $\rho_0 \in \mathcal{P}$. There exists a number $j_0 = j_0(\{\rho_j\}; R, \xi) > 0$ such that for all $j > j_0$ there exists a coupling of f_{ρ_j} and f_{ρ_0} such that the probability of $\Omega_3(f_{\rho_0}, f_{\rho_j}; R, \beta)$ outside*

$$\Omega_2(f_{\rho_0}; R, M) \cup \Omega_2(f_{\rho_j}; R, M)$$

is

$$\mathcal{P}(\Omega_3(f_{\rho_0}, f_{\rho_j}; R, \beta) \setminus (\Omega_2(f_{\rho_0}; R, M) \cup \Omega_2(f_{\rho_j}; R, M))) < \xi.$$

5.3. **Proof of Proposition 4.2.** For consistency with the earlier works the various events Ω_i in § 5.1 are defined in terms of properties of the relevant random fields imposed on balls of large radius; this is slightly inconsistent to the nodal counts $\mathcal{N}(\cdot;\cdot)$ in our main results that are defined on large squares. This however will not require any extra work due to the fortunate fact that the squares are contained in slightly bigger balls.

The following lemma states that, under the "stability assumption" on a function, its nodal components are stable.

Lemma 5.5 ([So], lemmas 6-7)**.** *Let $\beta > 0$ be a small number,*

$$\mathcal{D} = \mathcal{D}_{R+1} \subseteq \mathbb{R}^2$$

the side-$2(R+1)$ square, and $f \in C^1(B)$ be a smooth function on $\mathcal{D}$ such that

$$\min_{x \in \mathcal{D}} \max\{f(x), \|\nabla f(x)\|\} > \beta.$$

Suppose that $g \in C(\mathcal{D})$ is a continuous function on $\mathcal{D}$ such that

$$\sup_{x\in B} |g(x)| < \beta.$$

Then every nodal component γ of f lying entirely in $\mathcal{D}_R$ generates a unique nodal component $\widetilde{\gamma}$ of $(f+g)$ lying in $\mathcal{D}_{R+1}$ with distance $d(\gamma, \widetilde{\gamma}) < 1$ from $\widetilde{\gamma}$ (in fact, the stronger statement

$$\widetilde{\gamma} \subseteq \gamma_1 = \{x \in \mathcal{D}_{R+1} : \ d(x, \gamma) < 1\}$$

holds); different components of f correspond to different components of $(f + g)$.

Proof of Proposition 4.2. Let $R > 0$, $\xi > 0$, $\{\rho_j\} \subseteq \mathcal{P}$, and $\rho_0 \in \mathcal{P}$ be given. An application of Lemma 5.3, part 1 on $(\rho_0, R, \xi/4)$ and part 2 on $(\{\rho_j\}, \rho_0; R, \xi/8)$ yield a number $M > 0$ (a priori two different numbers that could be replaced by their maximum) such that both

$$\mathcal{P}(\Omega_2(\rho_0; R, M) < \frac{\xi}{8} \text{ and } \mathcal{P}(\Omega_2(\rho_j; R, M) < \frac{\xi}{8}, \tag{5.1}$$

for $j > j_0$ sufficiently big. An application of Lemma 5.2 on $(\rho_0, R, M, \xi/4)$ yields a number $\beta > 0$ sufficiently small so that

$$\mathcal{P}(\Omega_1(f_{\rho_0}; R, \beta) < \frac{\xi}{4} + \mathcal{P}(\Omega_2(f_{\rho_0}; R, M)) < \frac{\xi}{2}, \tag{5.2}$$

by (5.1). Finally, an application of Lemma 5.4 on $(\{\rho_j\}, \rho_0; R, M, \beta, \xi/4)$ yields a coupling of (f_{ρ_0}, f_{ρ_j}) such that for all $j > j_0$ we have

$$\mathcal{P}(\Omega_3(f_{\rho_0}, f_{\rho_j}; R, \beta)) < \frac{\xi}{4} + \mathcal{P}(\Omega_2(f_{\rho_0}; R, M)) + \mathcal{P}(\Omega_2(f_{\rho_j}; R, M)) < \frac{\xi}{2}, \tag{5.3}$$

again by (5.1).

Let

$$\Omega_0 := \Omega_1(f_{\rho_0}; R, \beta) \cup \Omega_3(f_{\rho_0}, f_{\rho_j}; R, \beta) \tag{5.4}$$

of probability

$$\mathcal{P}(\Omega_0) < \xi \tag{5.5}$$

by (5.2) and (5.3), provided that j is sufficiently big. On $\Omega \setminus \Omega_0$ the function f_{ρ_0} is stable in the sense that

$$\min_{x\in B(2R)} \max\{|f_{\rho_0}(x)|, \|\nabla f_{\rho_0}(x)\|\} \geq 2\beta, \tag{5.6}$$

and

$$\left\{\|f_{\rho_0} - f_{\rho_j}\|_{C^1(B(2R))} \leq \beta\right\}. \tag{5.7}$$

Together (5.6) and (5.7) imply the stability of f_{ρ_j}, i.e., that

$$\min_{x\in B(2R)} \max\{|f_{\rho_j}(x)|, \|\nabla f_{\rho_j}(x)\|\} \geq \beta \tag{5.8}$$

via the triangle inequality. Note that for $R \gg 0$ sufficiently big $\mathcal{D}_{R+1} \subseteq B(2R)$ so that all the above inequalities are satisfied in $\mathcal{D}_{R+1}$.

Now an application of Lemma 5.5 with $f = f_{\rho_0}$ and $g = f_{\rho_j} - f_{\rho_0}$, and upon bearing in mind (5.6) and (5.7) yields on $\Omega \setminus \Omega_0$ the r.h.s. of the inequality (4.1). The same argument now taking $f = f_{\rho_j}$ and $g = f_{\rho_0} - f_{\rho_j}$, this time employing (5.8) and (5.7) yields on $\Omega \setminus \Omega_0$ the l.h.s. of (4.1). The above shows that (4.1) holds on $\Omega \setminus \Omega_0$, and in addition (5.5) provided that j is sufficiently big. The proof of Proposition 4.2 is concluded.

□

5.4. **Proofs of the auxiliary lemmas 5.2-5.4.** We begin with the following simple lemma.

Lemma 5.6. *Let $\rho_1, \rho_2, \ldots \in \mathcal{P}$ be a sequence of spectral measures such that $\rho_k \Rightarrow \rho$, with the limiting measure $\rho \in \mathcal{P}$. Then*

$$\widehat{\rho_k}(\xi) \to \widehat{\rho}(\xi)$$

locally uniformly, i.e. $r_{\rho_k}(x) \to r_\rho(x)$, uniformly on compact subsets of $\mathbb{R}^2$. Moreover, the same holds for any (fixed) finite number of derivatives.

Proof. Let D be the closure of the support of the spectral measure; we recall the assumption that D is compact (this certainly holds for band-limited random waves, as well as for monochromatic waves.) Further, let $K \subset \mathbb{R}^2$ be compact. We note that the functions $\xi \to e(\xi \cdot x)$, as x ranges over elements in K is a uniformly continuous family. Moreover, as D is compact and we consider probability measures on D, we find that

$$\xi \to \widehat{\rho}(\xi) = \int_D e(\xi \cdot x) d\rho(x)$$

is uniformly continuous for all probability measures ρ on D, and that the Lipschitz estimate

$$|\widehat{\rho}(\xi) - \widehat{\rho}(\xi')| = O_D\left(|\xi - \xi'|\right)$$

holds for all ρ.

Let $\alpha > 0$ be given. Given $\xi \in K$, choose $k(\xi)$ such that $|\widehat{\rho_k}(\xi) - \widehat{\rho}(\xi)| < \alpha$ holds for all $k \geq k(\xi)$. Further, for each $\xi \in K$ there exists an open ball B_ξ centred at ξ such that

$$|\widehat{\rho_k}(\xi') - \widehat{\rho_k}(\xi)| < \alpha$$

for all $\xi' \in B_\xi$ and all k, and the same estimate holds for $\widehat{\rho}$.

As $\{B_\xi\}_{\xi \in K}$ is an open cover of the compact set K, we find that $K \subset \cup_{i=1}^{I} B_{\xi_i}$ for some finite collection of points $\xi_1, \ldots, \xi_I$. Define $k = \max\{k(\xi_i), i = 1, \ldots, I\}$. If $\xi \in K$ there exist i such that $\xi \in B_{\xi_i}$, and thus, for $l \geq k$,

$$|\widehat{\rho_l}(\xi) - \widehat{\rho}(\xi)| = |\widehat{\rho_l}(\xi) - \widehat{\rho_l}(\xi_i) + \widehat{\rho_l}(\xi_i) - \widehat{\rho}(\xi_i) + \widehat{\rho}(\xi_i) - \widehat{\rho}(\xi)| \leq 3\alpha$$

and hence the convergence is uniform in ξ. Finally, a similar argument gives that the same holds for a finite number of derivatives of r_{ρ_k}.

□

Proof of Lemma 5.2. The proof is very similar to the proof of [So, Lemma 5] presented in [So, p. 23]. The independence of $(f_\rho(x), \nabla f_\rho(x))$ in Lemma 3.1 as well as the determinant of $C(\rho)$ in (3.4) being bounded away from 0 play a crucial role at the end of the proof presented in [So, p. 23] in showing that both $f(x)$ and $\nabla f(x)$ being small is very rare. □

Proof of Lemma 5.3. The proof is very similar to the proof of [SW, Lemma 6.6]. Here to use the Sudakov-Fernique Comparison Inequality we invoke Lemma 5.6 so that the supremum of f_{ρ_j} and its derivatives over a compact domain is controlled by the supremum of f_{ρ_0} and its respective derivatives over the same domain.

□

Proof of Lemma 5.4. We employ [So, Lemma 4] which states that the conclusion of Lemma 5.4 holds if $r_{\rho_j} \to r_{\rho_0}$ locally uniformly together with their finitely many derivatives, i.e. that for all multi-index J, and $|J|$ bounded,

$$\sup_{\|x\| \leq 2R} |\partial_J r_{\rho_j}(x) - \partial_J r_{\rho_0}(x)| \to 0.$$

That this is so in our case follows from Lemma 5.6.

□

6. PROOF OF THEOREM 1.5: NODAL COUNT FOR ARITHMETIC RANDOM WAVES

6.1. **Proof of Theorem 1.5.** We begin by the following lemma asserting that the Nazarov-Sodin constant vanishes for the (tilted) Cilleruelo measure.

Lemma 6.1. *The Nazarov-Sodin constant of the Cilleruelo measure* (1.9) *vanishes, i.e.,*

$$c_{NS}(\nu_0) = 0.$$

Before proving Lemma 6.1 in § 6.2 we present the proof of Theorem 1.5.

Proof of Theorem 1.5 assuming Lemma 6.1. **Proof of part 1:** We use the natural quotient map $q : \mathbb{R}^2 \hookrightarrow \mathbb{T}^2$ and define the scaled random fields $g_n : \mathbb{R}^2 \to \mathbb{R}$ as

$$g_n(y) := f_n(q(y/\sqrt{n})).$$

Then g_n is a centred Gaussian random field with spectral measure μ_n on $\mathcal{S}^1$, as in (1.7) (one could also write $g_n = f_{\mu_n}$ though we will refrain from doing it to avoid confusion). It is then clear that the nodal domains of g_n lying inside the square $\mathcal{D}_{\sqrt{n}} = [-\sqrt{n}/2, \sqrt{n}/2]^2$ are in a $1-1$ correspondence with the nodal domains of f_n that do not intersect the image $q(\partial([-1/2, 1/2]^2)) \subseteq \mathbb{T}^2$ of the boundary of the fundamental domain of $\mathbb{T}^2$. Hence, under the notation of Lemma 3.2, we have

$$\left|\mathcal{N}(f_n) - \mathcal{N}(g_n; \sqrt{n}/2)\right| \le \mathcal{N}(g_n, \mathcal{C}_n), \tag{6.1}$$

where $\mathcal{C}_n$ is the boundary curve $\mathcal{C}_n = \partial[-\sqrt{n}/2, \sqrt{n}/2]$ of the side-$\sqrt{n}$ square. An application of Lemma 3.2 then yields

$$\mathcal{N}(f_{\mu_n}, \mathcal{C}_n) = O(\sqrt{n}).$$

This, together with (1.4) and (6.1) finally yields (1.11).

Proof of part 2:
Since $\mu \in \mathcal{P}_{symm}$ and μ is assumed to have no atoms, then μ satisfies the axioms $(\rho 1) - (\rho 3)$ of [So]. Lemma 5.6 then implies that, in the language of [So, Definition 1], the family $\{f_{n_j}\}$ of toral random fields has *translation invariant local limits* f_ρ. Hence [So, Theorem 4] implies (1.12) (see also [Ro, Theorem 1.2]).

Proof of part 3:
First, if μ is neither the Cilleruelo measure ν_0 in (1.9) nor the tilted Cilleruelo measure $\widetilde{\nu_0}$ in (1.10), then μ is supported on at least four distinct pairs of antipodal points. Thus, by [In, Remark 3] (or [So2]), $c_{NS}(\mu) > 0$. Conversely, the Nazarov-Sodin constant vanishes $c_{NS}(\nu_0) = c_{NS}(\widetilde{\nu_0}) = 0$ for both the Cilleruelo and tilted Cilleruelo measures by Lemma 6.1 (which is valid for $\widetilde{\nu_0}$ by rotation of $\pi/4$ symmetry).

Proof of part 4:
Let $\mathcal{B} \subseteq \mathcal{S}_{symm}$ be the set of weak-$*$ partial limits of $\{\mu_n\}$; we claim that $\mathcal{B}$ is *connected*. Once having the connectedness in our hands, part 4 of Theorem 1.5 follows from the continuity of c_{NS} (Theorem 1.3), vanishing $c_{NS}(\nu_0) = 0$ of the Nazarov-Sodin constant of the Cilleruelo measure (Theorem 1.5, part 3), and the positivity (1.3) of the universal Nazarov-Sodin constant.

To show the connectedness of $\mathcal{B}$ we recall that $\mathcal{B}$ is closed [KW2, Proposition 1.2] w.r.t. taking convolutions $\rho_1, \rho_2 \mapsto \rho_1 \star \rho_2$, and that there exists [KKW, Proposition 1.2] a path $a \mapsto \nu_a$, $a \in [0, \pi/4]$ between ν_0 the Cilleruelo measure and $\nu_{\pi/4} = \frac{d\theta}{2\pi}$ the uniform measure on $\mathcal{S}^1$; ν_a is the arc-length measure on $\theta \in [-a, a]$, symmetrised to be $\pi/2$-rotation invariant. The above implies that given $\rho \in \mathcal{B}$, we may construct a path

$$\{\rho \star \nu_a\}_{a \in [0, \pi/4]}$$

between ρ and $\rho \star \frac{d\theta}{2\pi} = \frac{d\theta}{2\pi}$, so that $\mathcal{B}$ is path-connected (in particular, connected).

□

6.2. **Proof of Lemma 6.1: the Nazarov-Sodin constant of the Cilleruelo measure vanishes.** We give two different proofs, each independently informative; the same proofs are valid for the tilted Cilleruelo. The first proof uses the fact that the limit random field can be realized explicitly as a trigonometric polynomial with only four nonzero coefficients. The second proof is based on a local computation of the number of "flips" in the direction of the line $x_1 = x_2$

Proof 1*: Limit random field.* Let

$$\nu_0 = \frac{1}{4}(\delta_{\pm 1} + \delta_{\pm i})$$

be the Cilleruelo measure; the corresponding covariance function is then

$$r_0(x) := \frac{1}{2}(\cos(x_1) + \cos(x_2)) \tag{6.2}$$

with $x = (x_1, x_2) \in \mathbb{R}^2$, and

$$\mathbb{E}[f_{\nu_0}(x) \cdot f_{\nu_0}(y)] = r_0(x - y),$$

for $x, y \in \mathbb{R}^2$. Let us describe the corresponding Gaussian random field $f_0 = f_{\nu_0}$ explicitly. We may realize it as

$$f_0(x) = \frac{1}{\sqrt{2}} \left(\xi_1 \cos(x_1) + \xi_2 \sin(x_1) + \xi_3 \cos(x_2) + \xi_4 \sin(x_2) \right), \tag{6.3}$$

where $(\xi_1, \xi_2, \xi_3, \xi_4)$ is a standard 4-variate Gaussian; equivalently $\{\xi_i\}_{i=1}^4$ are standard Gaussian i.i.d.

Alternatively, we may rewrite (6.3) as

$$f_0(x) = \frac{1}{\sqrt{2}} \cdot \left(a_1 \cdot \cos(x_1 + \eta_1) + a_2 \cdot \cos(x_2 + \eta_2) \right), \tag{6.4}$$

where $a_1 = \sqrt{\xi_1^2 + \xi_2^2}$, $a_2 = \sqrt{\xi_3^2 + \xi_4^2}$ are Rayleigh(1) distributed independent random variables (equivalently, χ with 2 degrees of freedom), and $\eta_1, \eta_2 \in [0, 2\pi)$ are random phases uniformly drawn in $[0, 2\pi)$. Let us now determine the zero set of f_0 in (6.4) on $\mathbb{T}^2$; we claim that f_0 has no compact nodal components at all; accordingly for every $R > 0$ we have

$$\mathcal{N}(f_0; R) \equiv 0.$$

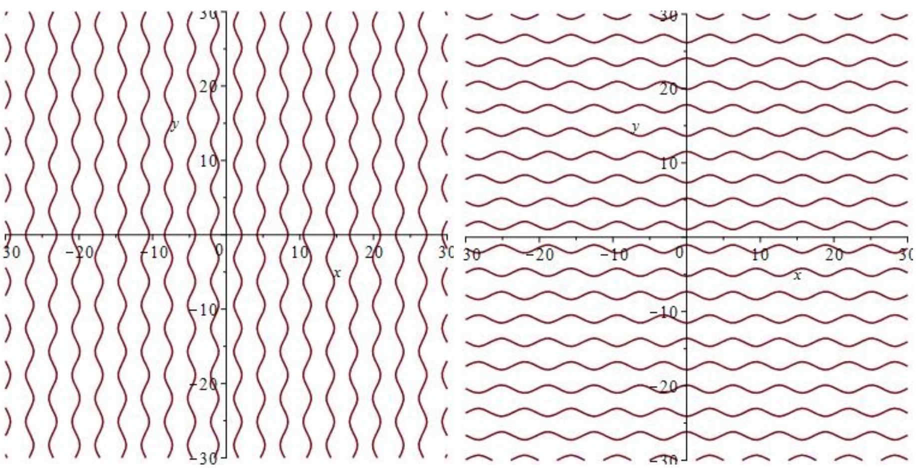

FIGURE 2. Solution plot for equations $2\cos(x_1) + \cos(x_2) = 0$ (left) and $\cos(x_1) + 2\cos(x_2) = 0$ (right). In both cases there are no compact nodal components.

First, by translation $y = x + (\eta_1, \eta_2)$, we may assume that $\eta_1 = \eta_2 = 0$, so that $f_0(x) = 0$ if and only if

$$a_1 \cdot \cos(y_1) = -a_2 \cdot \cos(y_2). \tag{6.5}$$

Now suppose that the coefficients in (6.5) satisfy $a_1 > a_2$ (occurring with probability $\frac{1}{2}$). Given y_1 there is a solution for y_2 to (6.5), if and only if

$$y_1 \in \left[\arccos\left(\frac{a_2}{a_1}\right), \pi - \arccos\left(\frac{a_2}{a_1}\right)\right] + k\pi \tag{6.6}$$

for some $k \in \mathbb{Z}$. A number y_1 lying in the open interval on the r.h.s. of (6.6) corresponds to precisely two solutions for y_2 in each period $y_2 \in [j \cdot 2\pi, (j+1) \cdot 2\pi)$ (depending on the parity of k in (6.6)). For the endpoints y_1 of the interval on the r.h.s. of (6.6) there exists a unique solution $y_2 = (2j+1)\pi$ and $2j\pi$ to the left and right endpoints respectively in case k in (6.6) is even, and the other way around in case k is odd. The above means that the solution curve of (6.5) consists of ascending oscillating periodic curves (see Figure 2, left) with no compact components at all. The situation when the coefficients in (6.5) satisfy $a_1 < a_2$ is a mirror image of the just considered (see Figure 2, right); the event $a_1 = a_2$ does almost surely not occur.

□

Proof 2: *Local estimates.* We reuse the notation (1.9) for the Cilleruelo measure ν_0, the covariance function $r_0(x)$ given by (6.2), and $f = f_{\nu_0}$; also recall the notational conventions that $f_1 = \partial_1 f = \partial f/\partial x_1$, $f_{12} = \partial_1\partial_2 f$ etc. Given a smooth *closed* planar curve $\gamma : [0, L] \to \mathbb{R}^2$ and a unit vector $\xi \in \mathcal{S}^1$ there exists a point $t \in [0, L]$ such that the tangent $\dot{\gamma}(t) = \pm\xi$ of γ is in the direction $\pm\xi$. Therefore

$$\mathbb{E}[\mathcal{N}(f; R)] \le \mathbb{E}[x \in \mathcal{D}_R : \ f_0(x) = f_1(x) + f_2(x) = 0]. \tag{6.7}$$

In what follows we will find that the r.h.s. of (6.7) vanish, and thus so does the l.h.s.; this certainly implies that $c_{NS}(\nu_0) = 0$.

To this end we define

$$F(x) := (f(x), f_1(x) + f_2(x))$$

and use Kac-Rice [AW, Theorem 6.3] to write

$$\mathbb{E}[x \in B(R) : \ f(x) = f_1(x) + f_2(x) = 0] = \int\limits_{B(R)} K_1(x)dx, \tag{6.8}$$

where

$$K_1(x) \equiv K_1(0) = \phi_{F(0)} \cdot \mathbb{E}[|J_F(0)| \big| F(0)]$$

by stationarity (see § 3); $F(0)$ is non-degenerate by the independence of the components of the Gaussian vector

$$(f(0), \nabla f(0)) \in \mathbb{R}^3.$$

Now

$$J_F(0) = \begin{pmatrix} f_1 & f_2 \\ f_{11} + f_{12} & f_{12} + f_{22} \end{pmatrix}, \tag{6.9}$$

where all of the matrix entries are evaluated at the origin. Moreover, a direct computation with r_0 reveals that

$$\mathrm{Var}(f_{12}) = 0,$$

i.e. $f_{12} = 0$ a.s. Hence (6.9) is

$$J_F(0) = \begin{pmatrix} f_1 & f_2 \\ f_{11} & f_{22} \end{pmatrix},$$

and

$$\det J_F(0) = f_1 \cdot f_{22} - f_2 \cdot f_{11};$$

conditioned on

$$f(0) = f_1 + f_2 = 0$$

this equals to

$$\det J_F(0) = f_1 \cdot f_{22} + f_1 \cdot f_{11} = f_1 \cdot (f_{11} + f_{22}) = -f_1 \cdot f = 0,$$

since f satisfies the Schrödinger equation $\Delta f + f = 0$, and we condition on $f(0) = 0$. Hence $K_1(x) \equiv 0$ vanishes identically, the expectation on the l.h.s. of (6.8) vanishes, which, as it was mentioned above, is sufficient to yield the statement of Theorem 1.5.

□

7. Spectral measures ρ with $d_{NS}(\rho) > 0$

We give two examples of trigonometric polynomials f, g, both realizable from the same spectral measure (with support on the three pairs of antipodal points $\{\pm(1,0), \pm(3,0), \pm(0,1)\}$), namely

$$f(x,y) = \sin(x) + 0.8 \cdot \sin(3x) + \sin(y), \quad g(x,y) = \sin(x) + 0.8 \cdot \sin(3x) + 0.2 \cdot \sin(y)$$

where f has many compact nodal domains, whereas g does not.

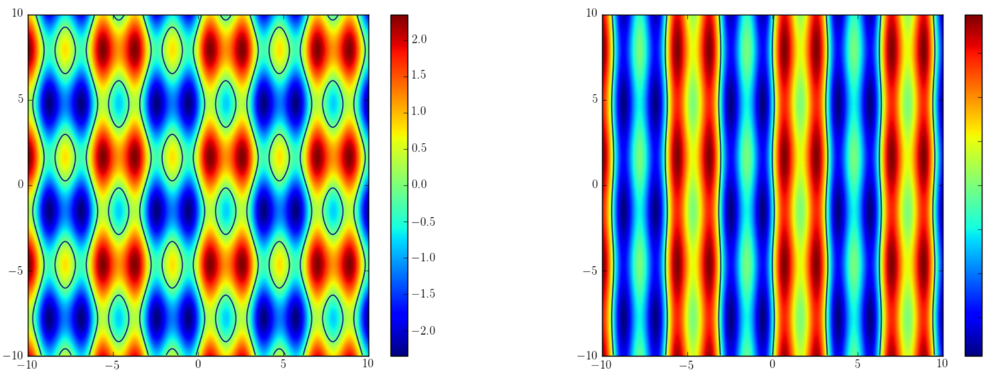

FIGURE 3. Plot of $f(x, y)$ (left) and $g(x, y)$ (right). Note that compact nodal components only occurs in the left plot. Nodal curves in black.

It is easy to verify that both f and g are stable in the sense that the density of the number of nodal domains (per area unit) remains the same under small perturbations of the form

$$f \to f + \epsilon_1 \sin(x) + \epsilon_2 \cos(x) + \epsilon_3 \sin(3x) + \epsilon_4 \cos(3x) + \epsilon_5 \sin(y) + \epsilon_6 \cos(y)$$

Thus both types of events (i.e., having no compact nodal domains or having a positive density of compact nodal domains per area unit) occur with probability > 0. Hence there exists ρ such that $d_{NS}(\rho) > 0$.

The spectral measure in the above example is not monochromatic, but using a recent result by M. Ingremeau we can give examples of monochromatic spectral measures ρ, also supported on three pairs of antipodal points, with the property that $d_{NS}(\rho) > 0$. Namely, let ρ be the uniform probability measure supported on the six points $\pm(1,0), \pm(0,1), \pm(1,1)/\sqrt{2}$. Letting

$$g(x,y) := 2\cos(x) + \cos(y)$$

we find that g has no compact nodal domains (cf. Figure 2); it is straightforward to verify that the gradient is non-vanishing on the nodal set of g. Since g is doubly periodic there exists $\beta > 0$ such that g has no

β-unstable points in $\mathbb{R}^2$ (cf. § 5.1). As g is stable, the nodal pattern persists for small perturbations of the form

$$g = f + \epsilon_1 \sin(x) + \epsilon_2 \sin(y) + \epsilon_3 \cos((x+y)/\sqrt{2}) + \epsilon_4 \sin((x+y)/\sqrt{2})$$

(for $\epsilon_1, \ldots, \epsilon_4$ sufficiently small). Hence, given any $\epsilon > 0$, there exists R_0 such that the event

$$\mathcal{N}(f_\rho; R)/(4R^2) < \epsilon,$$

for all $R \geq R_0$, occurs with positive probability. On the other hand, Ingremeau (cf. [In, Remark 3]) has shown that $c_{NS}(\rho) > 0$ for any spectral measure ρ with proper support on three or more pairs of antipodal points, and hence $d_{NS}(\rho) > 0$.

APPENDIX A. PLOTS OF CILLERUELO TYPE EIGENFUNCTION

We begin with a higher resolution plot of the eigenfunction shown in Figure 1. As can be seen the nodal domains tend to be either vertical or horizontal, and extend many wavelengths.

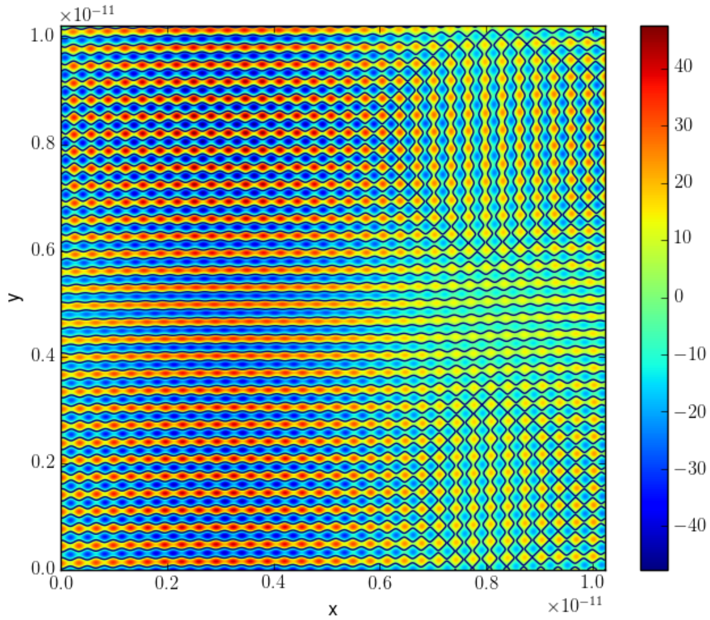

FIGURE 4. Fragment of a Cilleruelo type eigenfunction; nodal curves in black as before. Here $n = 9676418088513347624474653$ and $r_2(n) = 256$.

Below we give examples of the most extreme type of Cilleruelo eigenfunctions in terms of the spectral measure having smallest possible angular support. These arise from primes of the form $n = a^2 + 1$; we then have $r_2(n) = 8$ and the set of lattice points $\{(x, y) \in \mathbb{Z}^2 : x^2 + y^2 = n\}$ are of the form $\{(a, \pm 1), (-a, \pm 1), (1, \pm a), (-1, \pm a)\}$ and the angles between these vectors and either the x, or y, coordinate axis is very small for a large.

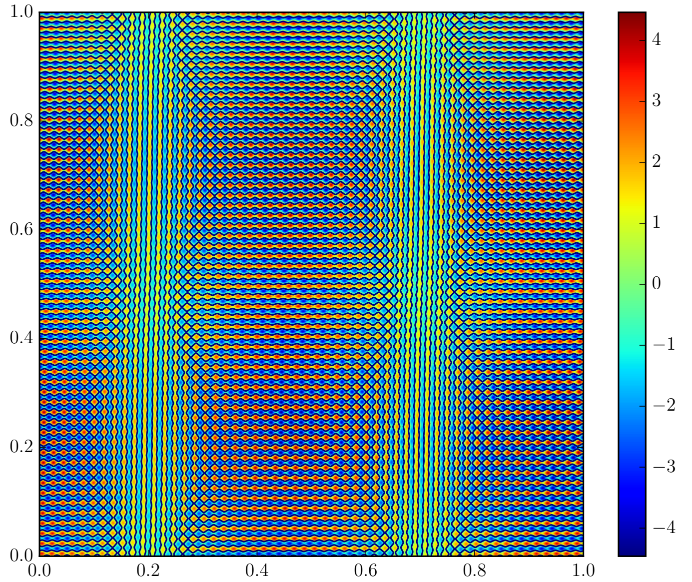

FIGURE 5. Plot of random Cilleruelo type eigenfunction, for $n = 54^2 + 1$ and $r_2(n) = 8$.

FIGURE 6. Plot of random Cilleruelo type eigenfunction, for $n = 54^2 + 1$ and $r_2(n) = 8$.

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