Degree project in mathematics

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Zooming Into the Apollonian Super-Packing

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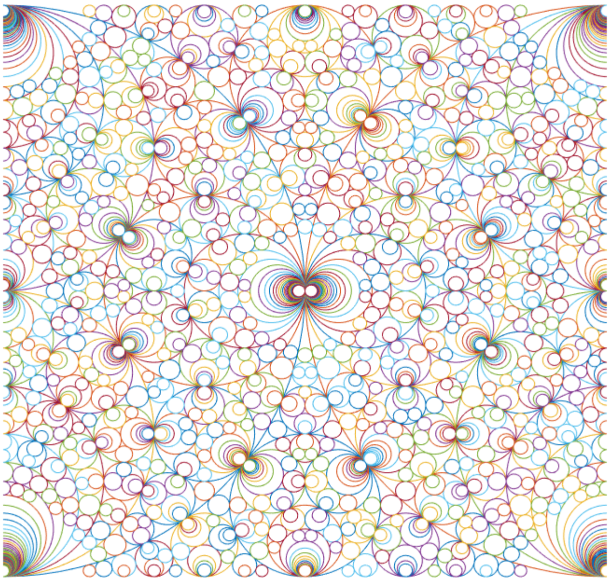
Abstract

The Apollonian super-packing is a remarkable arrangement of circles on the complex plane. For almost every point $z \in \mathbb{C}$, there is an infinite chain of circles surrounding it. In this thesis, we study consecutive circles in these chains and the ratios of consecutive curvatures $\gamma_{n-1}(z)/\gamma_n(z)$. Our method involves introducing a measure-preserving map T, which is conjectured to have a certain equidistribution property. From this assumption, we find an expression for the limiting distribution of $\gamma_{n-1}(z)/\gamma_n(z)$ as $n \to \infty$.



Contents

1 Introduction

Apollonian circle packings are classical configurations of mutually tangent circles obtained by a recursive process in which one starts with four mutually tangent circles, and then greedily fills the curvilinear triangles formed with more circles. As discovered by chemistry Nobel Prize laureate Frederick Soddy, who wrote a poem about the subject, if one chooses four circles of integer curvatures, then all other circles in the packing will also have integer curvatures [9]. Apollonian packings have since been thoroughly studied in both number theoretic and geometric lights. For instance, they are fractals of Hausdorff dimension approximately equal to 1.30568. [2].

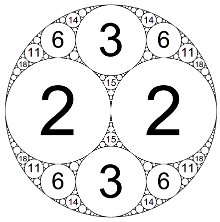


Figure 1: An example of an integral circle packing

This thesis studies a larger geometric object referred to as the Apollonian super-packing $\mathcal{P}_{\text{sup}}$, which in this article we realise as the orbit of the real line under the action of $SL(2, \mathbb{Z}[i])$. This is a collection of circles and lines, all with integer curvatures, as in regular Apollonian packings. In fact, the Apollonian super-packing contains a similar copy of every integral Apollonian packing. [4]

The Apollonian super-packing has been studied by Ronald L. Graham et al. [3] [4] [5]. The circles in it cover the complex plane in a dense manner, despite the fact that no two circles cross. Note that in other articles, the same object is referred to as the Gaussian Schmidt arrangement.

S. Chaubey et al. studied the Apollonian packing from a dynamic perspective and described how to construct complex continued fraction approximations using geodesic flow in hyperbolic 3-space [1]. They introduce a measure-preserving transformation which they conjecture to be ergodic. This connects the super-packing to the broader theory of Gaussian integer continued fractions, Farey octahedra, and the Picard group [7, 6].

The present thesis studies a related, but different, dynamical question. We study the set of all circles in $\mathcal{P}_{\text{sup}}$ surrounding a point z, which is chosen uniformly at random in the unit square $[0,1)+[0,1)i$. By enumerating their curvatures in increasing order, we produce a random sequence $\gamma_1(z), \gamma_2(z), \ldots$ of integers, the statistics of which encapsulates the hierarchical structure of the Apollonian Super Packing. In order to better understand the properties of this packing, we ask the question: What happens to the distribution of the ratios $\gamma_{n-1}(z)/\gamma_n(z)$ in the limit as $n \to \infty$?

Similarly to [1], we introduce a transformation T which preserves a measure μ. We then formulate an equidistribution conjecture, using which we prove that

$$\frac{\gamma_{n-1}(z)}{\gamma_n(z)}$$

converge in distribution to the probability measure on $[0, 1]$ with density

$$f(y) = \frac{3}{\pi} + \sum_{\alpha=2}^{\lfloor (y^2+1)/(2y) \rfloor} \frac{2N_\alpha}{\sqrt{\alpha^2 - 1}}, \qquad 0 < y \leq 1.$$

where $(N_\alpha)_{\alpha=1}^\infty$ is a counting sequence based on the Apollonian strip (see definition 6.5). The thesis is organized as follows. Section 2 recalls the necessary background on Apollonian packings, generalized circles, Möbius transformations, hyperbolic area, and the modular group, and introduces the Apollonian super-packing together with the semi-group Ω. Section 3 develops the geometry of Möbius transformations: it derives explicit centre–radius formulas for images of $\widehat{\mathbb{R}}$ under elements of $SL(2, \mathbb{Z}[i])$. Section 4 studies the structure of the super-packing itself, proving the non-crossing property, organizing the disks surrounding a point into chains, and describing the atoms of Ω. Section 5 constructs the map T on the quotient space $\mathbb{M}$, proves that it is well-defined and bijective almost everywhere, and shows that it preserves the measure μ. Section 6 relates the dynamics of T to successive curvature ratios, states the equidistribution conjecture, and derives the limiting distribution conditional on it.

2 Background

Four circles to the kissing come.
The smaller are the benter.
The bend is just the inverse of
The distance form the center.
Though their intrigue left Euclid dumb
There's now no need for rule of thumb.
Since zero bend's a dead straight line
And concave bends have minus sign,
The sum of the squares of all four bends
Is half the square of their sum.

—F. Soddy, *The Kiss Precise*

Integral Apollonian packings Consider four mutually touching circles ω_1, ω_2, ω_3, and ω_4, a configuration known as a Descartes configuration. The geometry of Descartes configurations has interested mathematicians dating back to the ancient Greeks, however it was Descartes who found an algebraic equation relating their curvatures. He showed the following: If a, b, c, d are the signed curvatures of the four circles, signed in the sense that a curvature is taken to be negative if and only if that circle contains the other circles internally, then [8]

$$2(a^2 + b^2 + c^2 + d^2) = (a + b + c + d)^2.$$

Using this theorem, one can show that any Descartes quadruple with integer curvatures can be extended to an entire Apollonian circle packing with integer curvatures.

The Riemann sphere and generalized circles. Let

$$\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}, \quad \widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}. \quad \mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}.$$

A generalized circle in $\widehat{\mathbb{C}}$ means either an ordinary Euclidean circle in $\mathbb{C}$ or a Euclidean straight line together with the point ∞. Thus, a line is treated as a circle passing through ∞. Similarly, a generalised disk is any of the complementary components of a generalised circle. The curvature of a circle or disk is the reciprocal of its radius. In this thesis, we refer to ordinary (bounded) circles and disks as "genuine".

Möbius transformations. Recall that for any commutative ring R, $SL(2,R)$ denotes the set of all 2×2 matrices A for which $\det(A)=1$.

A Möbius transformation is a map of $\widehat{\mathbb{C}}$ of the form

$$z \longmapsto \frac{az+b}{cz+d}, \qquad ad-bc\neq 0.$$

It is understood that $z=-d/c$ is sent to ∞ when $c\neq 0$, and that ∞ is sent to a/c when $c\neq 0$ and to ∞ when $c=0$.

Each matrix $A=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL(2,\mathbb{C})$ can be identified with a corresponding Möbius transformation $\frac{az+b}{cz+d}$, which we denote by $A(z)$. Scalar multiples of the same matrix define the same transformation. With this notation, matrix multiplication agrees with composition:

$$(AB)(z)=A(B(z)).$$

For

$$A=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL(2,\mathbb{C}),$$

the elementary identities

$$A'(z)=\frac{1}{(cz+d)^2}, \qquad A(z)-A(w)=\frac{z-w}{(cz+d)(cw+d)}$$

hold away from the poles. These identities are repeatedly used below.

Möbius transformations send generalized circles to generalized circles. In particular, if $A\in SL(2,\mathbb{C})$, then $A(\widehat{\mathbb{R}})$ is a generalized circle, and $A(\mathbb{H})$ is one of the two components of $\widehat{\mathbb{C}}\setminus A(\widehat{\mathbb{R}})$. The next section gives the explicit centre-radius formula for this component in the cases relevant here.

The upper half-plane and hyperbolic area. We write

$$\mathbb{H}=\{z\in\mathbb{C}:\mathrm{Im}z>0\}, \qquad \mathbb{H}^-=\{z\in\mathbb{C}:\mathrm{Im}z<0\}.$$

If

$$M=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL(2,\mathbb{R}),$$

then

$$\mathrm{Im}(Mz)=\frac{\mathrm{Im}z}{|cz+d|^2}.$$

Thus $SL(2,\mathbb{R})$ preserves both $\mathbb{H}$ and $\mathbb{H}^-$, and it preserves the hyperbolic area element

$$d\mu_{\mathbb{H}}(z)=\frac{d^2z}{(\mathrm{Im}z)^2}, \qquad d^2z=dx\,dy \quad (z=x+iy).$$

The modular group and a fundamental domain. The subgroup

$$\Gamma=SL(2,\mathbb{Z})$$

will play a special role. It acts on $\mathbb{H}$ by Möbius transformations. Since the entries are real and the determinant is one, the formula

$$\mathrm{Im}(\gamma z)=\frac{\mathrm{Im}z}{|cz+d|^2}, \qquad \gamma=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL(2,\mathbb{R}),$$

shows that every $\gamma \in \Gamma$ maps $\mathbb{H}$ to itself. The matrices I and $-I$ induce the same Möbius transformation, so the action of Γ on $\mathbb{H}$ has kernel $\{\pm I\}$. Thus, as a group of transformations, Γ is naturally identified with $PSL(2,\mathbb{Z}) = \Gamma/\{\pm I\}$. We nevertheless work with Γ, since this is more convenient when comparing it with subsets of $SL(2,\mathbb{Z}[i])$.

A fundamental domain of Γ is a domain containing exactly one point from every Γ-orbit. In this thesis, we will fix

$$\mathcal{F} = \left\{ z \in \mathbb{H} : |\mathrm{Re} z| \leq \frac{1}{2},\ |z| \geq 1 \right\}.$$

This region is not a strict fundamental domain in the pointwise sense, because some boundary points are identified. The vertical sides are identified by U, and the circular arc $|z| = 1$ is identified by S. For integration and measure-theoretic arguments, this distinction is irrelevant, since the boundary has hyperbolic measure zero. The hyperbolic area of the fundamental domain is

$$\begin{aligned}\mu_{\mathbb{H}}(\mathcal{F}) &= \int_{-1/2}^{1/2} \int_{\sqrt{1-x^2}}^{\infty} \frac{dy\, dx}{y^2} \\ &= \int_{-1/2}^{1/2} \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{3}.\end{aligned}$$

By complex conjugation, the action of Γ on the lower half-plane $\mathbb{H}^-$ has the corresponding fundamental domain

$$\mathcal{F}^- = \{\overline{z} : z \in \mathcal{F}\},$$

and $\Gamma\backslash\mathbb{H}^-$ also has hyperbolic area $\pi/3$.

3 Geometry of Möbius transformations

Before we start looking into the structure of the Apollonian super-packing, we set a geometric foundation regarding circles and möbius transformations.

Definition 3.1. For any matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C}),$$

let

$$\begin{aligned}\alpha(A) &= \mathrm{Re}(a\bar{d} - b\bar{c}), \\ \beta(A) &= \mathrm{Im}(a\bar{d} - b\bar{c}), \\ \gamma(A) &= 2\mathrm{Im}(d\bar{c}), \\ \delta(A) &= 2\mathrm{Im}(b\bar{a}).\end{aligned}$$

The factors of 2 ensure that $\gamma(A)$ and $\delta(A)$ are even integers when A has Gaussian-integer entries.

Proposition 3.2. *For any $A \in SL(2,\mathbb{C})$,*

$$\alpha(A)^2 + \beta(A)^2 - \gamma(A)\delta(A) = 1$$

Proof. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C}), \qquad P = A\overline{A}^{-1},$$

where $\overline{A}$ denotes entrywise complex conjugation.

Since

$$\overline{A}^{-1} = \begin{pmatrix} \overline{d} & -\overline{b} \\ -\overline{c} & \overline{a} \end{pmatrix},$$

we get

$$P = A\overline{A}^{-1} = \begin{pmatrix} a\overline{d} - b\overline{c} & b\overline{a} - a\overline{b} \\ c\overline{d} - d\overline{c} & d\overline{a} - c\overline{b} \end{pmatrix} = \begin{pmatrix} \alpha(A) + \beta(A)i & \delta(A)i \\ -\gamma(A)i & \alpha(A) - \beta(A)i \end{pmatrix}.$$

The proposition then follows from the observation that $\det(P) = 1$. □

Proposition 3.3. *For any $A \in SL(2, \mathbb{C})$ and any $X \in SL(2, \mathbb{R})$,*

$$\alpha(A) = \alpha(AX), \quad \beta(A) = \beta(AX), \quad \gamma(A) = \gamma(AX), \quad \delta(A) = \delta(AX)$$

Proof. As before, we have

$$A\overline{A}^{-1} = \begin{pmatrix} \alpha(A) + \beta(A)i & \delta(A)i \\ -\gamma(A)i & \alpha(A) - \beta(A)i \end{pmatrix}.$$

Thus, for any $A, B \in SL(2, \mathbb{C})$ for which $A\overline{A}^{-1} = B\overline{B}^{-1}$, we have $\alpha(A) = \alpha(B)$, $\beta(A) = \beta(B)$, $\gamma(A) = \gamma(B)$ and $\delta(A) = \delta(B)$. Indeed, by setting $B = AX$, we get

$$B\overline{B}^{-1} = AX\overline{(AX)}^{-1} = AXX^{-1}\overline{A}^{-1} = A\overline{A}^{-1}$$

as desired. □

Proposition 3.4. *Let*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C}),$$

and write

$$\alpha = \alpha(A), \quad \beta = \beta(A), \quad \gamma = \gamma(A), \quad \delta = \delta(A).$$

If $\gamma = 0$, then $A(\widehat{\mathbb{R}})$ is the line $2\beta x - 2\alpha y + \delta = 0$. Otherwise, $A(\widehat{\mathbb{R}})$ is the circle with centre

$$z_0 = \frac{-\beta + \alpha i}{\gamma}$$

and radius

$$r = \frac{1}{|\gamma|}.$$

If $\gamma > 0$, then $A(\mathbb{H})$ is its interior.

Proof. By proposition 3.2,

$$\alpha^2 + \beta^2 - \gamma\delta = 1$$

and

$$A\overline{A}^{-1} = \begin{pmatrix} \alpha + \beta i & \delta i \\ -\gamma i & \alpha - \beta i \end{pmatrix}.$$

Now

$$z \in A(\widehat{\mathbb{R}}) \iff A^{-1}(z) = \overline{A^{-1}(z)} \iff z = A\overline{A}^{-1}(\overline{z}).$$

Writing $z = x + iy$, this equation becomes

$$z = \frac{(\alpha + \beta i)\overline{z} + \delta i}{-\gamma i\,\overline{z} + \alpha - \beta i},$$

or equivalently

$$\gamma(x^2 + y^2) + 2\beta x - 2\alpha y + \delta = 0.$$

If $\gamma = 0$, we get $2\beta x - 2\alpha y + \delta = 0$ as desired. Otherwise, completing the square and using $\alpha^2 + \beta^2 - \gamma\delta = 1$, we obtain

$$\left(x + \frac{\beta}{\gamma}\right)^2 + \left(y - \frac{\alpha}{\gamma}\right)^2 = \frac{1}{\gamma^2}.$$

Thus $A(\widehat{\mathbb{R}})$ is the circle with centre

$$\frac{-\beta + \alpha i}{\gamma}$$

and radius

$$\frac{1}{|\gamma|}.$$

Finally, if $\gamma > 0$, then $c \neq 0$ and

$$\gamma = \frac{d\overline{c} - c\overline{d}}{i} = 2|c|^2 \mathrm{Im}\left(\frac{d}{c}\right).$$

Hence

$$\mathrm{Im}(A^{-1}(\infty)) = \mathrm{Im}\left(-\frac{d}{c}\right) < 0.$$

So ∞ lies in the image of the lower half-plane. Therefore $A(\mathbb{H})$ is the component of $\widehat{\mathbb{C}} \setminus A(\widehat{\mathbb{R}})$ not containing ∞, namely the interior of the circle. $\square$

4 Super-apollonian circles and disks

We will now look into the structure of the Apollonian super-packing. Let

$$\mathbb{Z}[i] = \{m + ni : m, n \in \mathbb{Z}\}$$

be the ring of Gaussian integers. The Apollonian super-packing (shown in Figure 2) is then given by

$$\mathcal{P}_{\mathrm{sup}} = \{A(\hat{\mathbb{R}}) \ : \ A \in SL(2, \mathbb{Z}[i])\}$$

For our purposes, we are concerned with a closely related object, namely, the semi-group

$$\Omega = \left\{A \in SL(2, \mathbb{Z}[i]) \ : \ A(\mathbb{H}) \subset \mathbb{H}\right\}$$

and the collection of generalized disks

$$\mathcal{D} = \{A(\mathbb{H}) \ : \ A \in \Omega\}$$

The set of genuine disks in $\mathcal{D}$ is exactly the interiors of the circles in the Apollonian super-packing on the upper half plane.

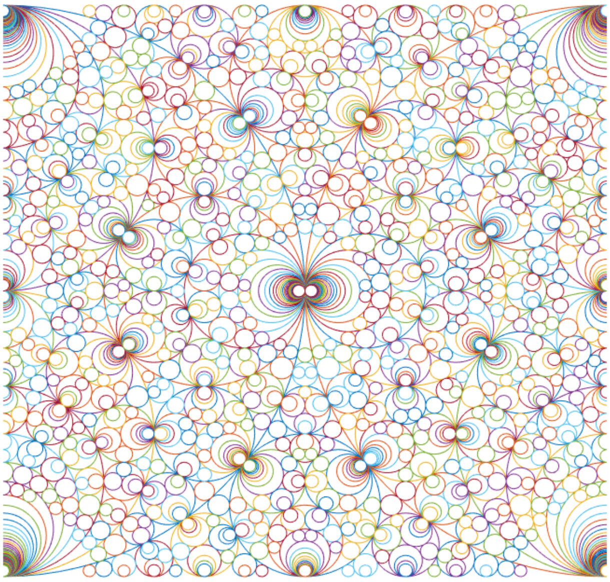


Figure 2: All circles in the Apollonian super packing within the unit square $[0,1]+[0,1]i$, with curvature less than 50. Generated with MATLAB.

Proposition 4.1. *If $A, B \in SL(2,\mathbb{Z}[i])$, then the generalized circles $A(\widehat{\mathbb{R}})$ and $B(\widehat{\mathbb{R}})$ never intersect in exactly two points.*

Proof. It suffices to prove that for any $X \in SL(2,\mathbb{Z}[i])$, the curve $X(\widehat{\mathbb{R}})$ does not cross the real line $\widehat{\mathbb{R}}$ in two points. Indeed, if $X = A^{-1}B$, then

$$A(\widehat{\mathbb{R}}) \cap B(\widehat{\mathbb{R}}) = A\big(\widehat{\mathbb{R}} \cap X(\widehat{\mathbb{R}})\big),$$

and A is a bijection of $\widehat{\mathbb{C}}$.

So let

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}[i])$$

Since $a, b, c, d \in \mathbb{Z}[i]$, the products in the definitions of $\alpha(X), \beta(X), \gamma(X)$, and $\delta(X)$ are Gaussian integers. Hence

$$\alpha(X), \beta(X), \gamma(X), \delta(X) \in \mathbb{Z}.$$

In fact, $\gamma(X)$ and $\delta(X)$ are even. Moreover,

$$\alpha(X) + \beta(X)i = a\overline{d} - b\overline{c}, \qquad \delta(X)i = b\overline{a} - a\overline{b}, \qquad -\gamma(X)i = c\overline{d} - d\overline{c}$$

and complex conjugation is trivial modulo 2: for every $z \in \mathbb{Z}[i]$,

$$z - \overline{z} \in 2\mathbb{Z}[i].$$

Hence

$$\alpha(X) + \beta(X)i = a\overline{d} - b\overline{c} \equiv ad - bc = 1 \pmod 2.$$

In other words, $\alpha(X)$ is odd and $\beta(X)$ is even.

If $\gamma(X) = 0$, then proposition 3.4 shows that $X(\widehat{\mathbb{R}})$ is the line

$$2\beta(X)x - 2\alpha(X)y + \delta(X) = 0.$$

Since $\gamma(X) = 0$, the identity $\alpha(X)^2 + \beta(X)^2 - \gamma(X)\delta(X) = 1$ becomes

$$\alpha(X)^2 + \beta(X)^2 = 1.$$

As $\alpha(X)$ is odd and $\beta(X)$ is even, this forces $\beta(X) = 0$ and $\alpha(X) = \pm 1$. Hence $X(\widehat{\mathbb{R}})$ is the horizontal line

$$y = \frac{\delta(X)}{2\alpha(X)}.$$

Therefore $X(\widehat{\mathbb{R}})$ is either equal to $\widehat{\mathbb{R}}$, or it intersects $\widehat{\mathbb{R}}$ only at ∞. In particular, the intersection is not exactly two points.

Now assume $\gamma(X) \neq 0$. Then proposition 3.4 shows that $X(\widehat{\mathbb{R}})$ is the circle with centre

$$z_0 = \frac{-\beta(X) + \alpha(X)i}{\gamma(X)}$$

and radius

$$r = \frac{1}{|\gamma(X)|}.$$

Its distance to the real line is

$$|\mathrm{Im}(z_0)| = \frac{|\alpha(X)|}{|\gamma(X)|}.$$

Since $\alpha(X)$ is odd, we have $|\alpha(X)| \geq 1$, so

$$|\mathrm{Im}(z_0)| \geq \frac{1}{|\gamma(X)|} = r.$$

Thus, the circle is either tangent to the real line or disjoint from it, and again the intersection cannot consist of exactly two points. $\square$

Corollary 4.2. *If $A, B \in \Omega$, then at least one of the following is true:*

(i) $A(\mathbb{H}) \subset B(\mathbb{H})$

(ii) $B(\mathbb{H}) \subset A(\mathbb{H})$

(iii) $A(\mathbb{H}) \cap B(\mathbb{H}) = \varnothing$.

Proof. If two generalised disks overlap with neither containing the other, their boundaries must cross in exactly two points on $\hat{\mathbb{C}}$, contradicting proposition 4.1. $\square$

Let

$$\mathcal{D} = \{A(\mathbb{H}) : A \in \Omega\}$$

be the collection of all Ω-domains. For $z \in \mathbb{H}$, define

$$\mathcal{C}(z) = \{D \in \mathcal{D} : z \in D\}.$$

Lemma 4.3. *For every $z \in \mathbb{H}$, the collection $\mathcal{C}(z)$ is linearly ordered by inclusion, producing the (either infinite or finite) enumeration*

$$D_0(z) \supsetneq D_1(z) \supsetneq D_2(z) \supsetneq \cdots,$$

where $D_0(z) = \mathbb{H}$ and each $D_{n+1}(z)$, when it exists, is the largest proper element of $\mathcal{C}(z)$ contained in $D_n(z)$.

Proof. By Corollary 4.2, two domains in $\mathcal{C}(z)$ cannot overlap without one containing the other. Since every domain in $\mathcal{C}(z)$ contains z, the family $\mathcal{C}(z)$ is therefore linearly ordered by inclusion.

Now, let $D \in \mathcal{C}(z)$. We prove the existence of the enumeration by showing that there is only a finite number of disks and half-planes in $\mathcal{C}(z)$ containing D.

Using 3.4 and corollary 4.2, one can show that all half-planes in $\mathcal{C}(z)$ are all of the form $\{w : \mathrm{Im}(w) > n\}$ for some integer $0 \le n < \lceil \mathrm{Im}(z) \rceil$. The number of such half-planes is at most $\lceil \mathrm{Im}(z) \rceil$, and in particular, finite. Now consider the genuine disks. Genuine disks in $\mathcal{C}(z)$ all have positive integer curvatures and thus, the number of genuine disks containing D is bounded by the curvature of D. We conclude that the number of domains containing D is finite, thus proving the lemma. $\square$

Let

$$\Lambda = \{z \in \mathbb{H} : |\mathcal{C}(z)| = \infty\}.$$

For $z \in \Lambda$, let $\gamma_n(z)$ denote the curvature of $D_n(z)$. An atom of a monoid M is a noninvertible element that cannot be written as a product of two noninvertible elements of M. Let $\mathcal{A}$ denote the set of atoms of Ω. Let

$$\mathcal{D}_1 = \{A(\mathbb{H}) : A \in \mathcal{A}\}$$

be the collection of atom domains. The boundaries of the disks in $\mathcal{D}_1$ is shown in figure 3. Taking the closure of their union gives a shape known as an Apollonian strip.

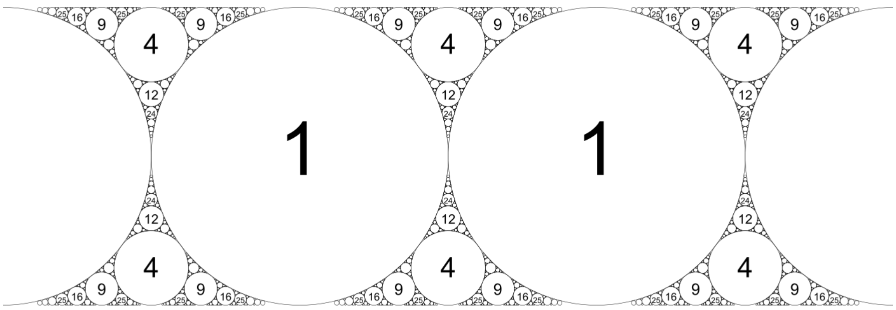


Figure 3: The Apollonian strip.

Lemma 4.4. *Distinct atom domains are disjoint.*

Proof. Let $D = X(\mathbb{H})$ and $D' = Y(\mathbb{H})$ be atom domains. Suppose that they are different but not disjoint. By Corollary 4.2, either $D \subsetneq D'$ or $D' \subsetneq D$. Assume $D \subsetneq D'$. Then

$$Y^{-1}X(\mathbb{H}) \subsetneq \mathbb{H},$$

so $Y^{-1}X$ is a noninvertible element of Ω. Hence $X = Y(Y^{-1}X)$ factors the atom X into two noninvertible elements of Ω, a contradiction. The other containment is identical. $\square$

Proposition 4.5. *The set $\mathbb{H} \setminus \Lambda$ has hyperbolic measure of zero.*

Proof. This follows from the observation of Graham et al., namely that points lying in the interior of an infinite nested sequence of circles in the super-packing have full Lebesgue measure [4]. $\square$

Lemma 4.6. *The invertible elements of Ω are precisely Γ. Moreover, if $A, B \in SL(2, \mathbb{Z}[i])$ and $A(\mathbb{H}) = B(\mathbb{H})$, then $A^{-1}B \in \Gamma$.*

Proof. Every element of Γ preserves both $\mathbb{H}$ and $\mathbb{H}^-$, so it is invertible in Ω. Conversely, let $G \in SL(2, \mathbb{Z}[i])$ satisfy $G(\mathbb{H}) = \mathbb{H}$. Then $G(\widehat{\mathbb{R}}) = \widehat{\mathbb{R}}$. Since $G(\widehat{\mathbb{R}}) = \widehat{\mathbb{R}}$, the equation from proposition 3.4,

$$\gamma(G)(x^2 + y^2) + 2\beta(G)x - 2\alpha(G)y + \delta(G) = 0,$$

is the equation $y = 0$. Hence $\gamma(G) = \beta(G) = \delta(G) = 0$ and $\alpha(G)^2 = 1$.

If $\alpha(G) = 1$, then $G\overline{G}^{-1} = I$, so $G = \overline{G}$ and all entries of G are rational integers. Thus $G \in \Gamma$. If $\alpha(G) = -1$, then $G\overline{G}^{-1} = -I$, so $G = -\overline{G}$ and $G = iR$ for some real integral matrix R. Since $\det G = 1$, we have $\det R = -1$. Multiplication of a matrix by the scalar i does not change its Möbius transformation, so G acts as the real matrix R. For a real matrix $R = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$$\mathrm{Im}\left(\frac{az + b}{cz + d}\right) = \frac{\det(R)\mathrm{Im}z}{|cz + d|^2}.$$

Since $\det R = -1$, this sends $\mathbb{H}$ to $\mathbb{H}^-$. This contradicts $G(\mathbb{H}) = \mathbb{H}$. Therefore $G \in \Gamma$.

Finally, if $A(\mathbb{H}) = B(\mathbb{H})$, then $A^{-1}B(\mathbb{H}) = \mathbb{H}$, so $A^{-1}B \in \Gamma$ by the first part. $\square$

Definition 4.7. For $A \in SL(2, \mathbb{Z}[i])$, set

$$A^* = \overline{A}^{-1}.$$

Lemma 4.8. *If $A \in \Omega$, then $A^* \in \Omega$.*

Proof. Since A^{-1} has Gaussian-integer entries and conjugation preserves $\mathbb{Z}[i]$, we have $A^* \in SL(2, \mathbb{Z}[i])$. Let $\iota(z) = \overline{z}$. As a map of $\widehat{\mathbb{C}}$,

$$A^* = \iota \circ A^{-1} \circ \iota.$$

If $z \in \mathbb{H}$, then $\iota(z) \in \mathbb{H}^-$. Since $A(\mathbb{H}) \subset \mathbb{H}$, the lower half-plane lies in the complementary component of $\widehat{\mathbb{C}} \setminus A(\widehat{\mathbb{R}})$ different from $A(\mathbb{H})$. Therefore $A^{-1}(\iota(z)) \in \mathbb{H}^-$. Applying ι once more gives $A^*(z) \in \mathbb{H}$, so $A^* \in \Omega$. $\square$

The operation $A \mapsto A^*$ also satisfies $(BC)^* = C^*B^*$. Therefore, it also preserves invertibility in Ω, and a factorization of A^* into two noninvertible elements of Ω is equivalent, after applying $A \mapsto A^*$, to a factorization of A into two noninvertible elements of Ω. Thus, A is an atom if and only if A^* is an atom.

Lemma 4.9. *The set Λ is invariant under Γ. Moreover, each $z \in \Lambda$ lies in a unique atom domain $D \in \mathcal{D}_1$, and if $D = X(\mathbb{H})$ with $X \in \mathcal{A}$, then*

$$D \cap \Lambda = X(\Lambda).$$

Hence

$$\Lambda = \bigsqcup_{D \in \mathcal{D}_1} (D \cap \Lambda).$$

Proof. Since every element of Γ is invertible in Ω, applying an element of Γ to the chain of z gives the chain of its image. Thus Λ is Γ-invariant.

Let $z \in \Lambda$. The first proper element $D_1(z)$ of its chain has the form $X(\mathbb{H})$ for some $X \in \Omega$. The element X is an atom. Indeed, if $X = YZ$ with Y, Z noninvertible in Ω, then

$$X(\mathbb{H}) = Y(Z(\mathbb{H})) \subsetneq Y(\mathbb{H}) \subsetneq \mathbb{H},$$

contradicting the definition of $D_1(z)$. Thus z lies in an atom domain. Distinct atom domains are disjoint by Lemma 4.4, so this atom domain is unique.

Now let $D = X(\mathbb{H})$ be an atom domain. If $u \in \Lambda$, then pushing the infinite chain of u forward by X gives an infinite chain through $X(u)$, so $X(\Lambda) \subset D \cap \Lambda$. Conversely, if $z \in D \cap \Lambda$, then D is the first proper element of the chain of z. Pulling back the tail of that chain under X gives an infinite chain through $X^{-1}(z)$, so $X^{-1}(z) \in \Lambda$. Hence $z \in X(\Lambda)$. $\square$

5 Dynamic formulation

In this section, we introduce the measure-preserving transformation T. In the next section, we will show how it is connected with the disk chains $\mathcal{C}(z)$ from the previous section.

Definition 5.1. Put

$$\Lambda^- = \{\overline{z} : z \in \Lambda\}. \quad \mathbb{H}^-_* = \mathbb{H}^- \cup \mathbb{R} \cup \{\infty\}.$$

For $w, w' \in \mathbb{H}^-_*$ and $z, z' \in \Lambda$, write $(w, z) \sim (w', z')$ if there is a $\gamma \in \Gamma$ such that

$$w' = \gamma(w), \qquad z' = \gamma(z),$$

where the action on the first coordinate is the usual action on the Riemann sphere. Define the quotient space

$$\mathbb{M} = \Gamma \backslash (\mathbb{H}^-_* \times \Lambda),$$

and its full-measure subset

$$\mathbb{M}_\Lambda = \Gamma \backslash (\Lambda^- \times \Lambda) \subset \mathbb{M}.$$

The quotient space $\mathbb{M}$ admits two useful geometric interpretations. One illuminates connections with geodesic flow, and the other shows how T relates to our original curvature quotient question.

The first interpretation is to regard (w, z) as the endpoints of a geodesic g in the upper half-space model of hyperbolic 3-space. The condition $\mathrm{Im}(w) < 0$, $\mathrm{Im}(z) > 0$ then guarantees that g intersects the hyperbolic 3-plane above the extended real line, which we may denote by p_0. If $z \in \Lambda$, then this geodesic will also intersect another half-plane p_1 whose boundary is also a super-Apollonian circle. The transformation T is then an isometry that moves p_1 to p_0. By imposing that $T \in SL(2, \mathbb{Z}i)$, then this isometry is unique up to left multiplication by Γ.

The second interpretation is to regard (w, z) as parametrizing an Apollonian packing G (specified up to similarity) and a point p inside the convex hull of G. We set G to be any Apollonian packing $A(S)$ where S is the standard Apollonian strip and A is an arbitrary Möbius transformation sending w to ∞ (all such packings are similar since the only Möbius transformations that preserve the point at infinity are similarity transformations). Then, we set $p = A(z)$. The condition $\mathrm{Im}(w) < 0$ then guarantees that $A(\mathbb{H})$ is a genuine disk, and the condition $\mathrm{Im}(z) > 0$ keeps p in $A(\mathbb{H})$. If $z \in \Lambda$, then p is surrounded by a circle $\omega \in G$ other than $A(\hat{\mathbb{R}})$. Inverting G through ω using circle inversion then gives another packing-point pair, and this is the action performed by $T(w, z)$.

We now define the transformation T precisely.

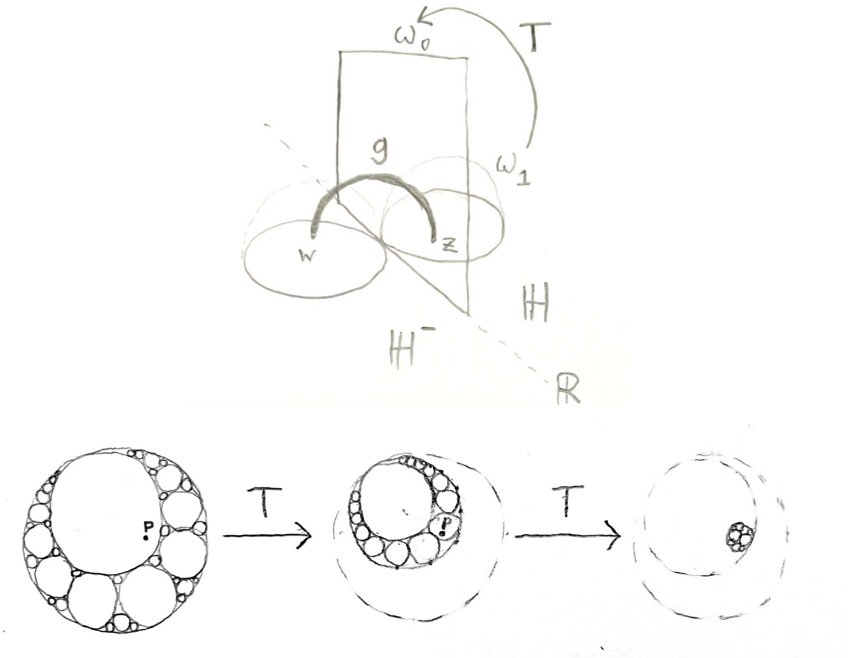


Figure 4: Two interpretations of (w, z). The first image illustrates the geodesic interpretation, and the second image illustrates the circle-point pair interpretation.

Definition 5.2. Let $[(w, z)] \in \mathbb{M}$. By Lemma 4.9, there is a unique atom domain $D \in \mathcal{D}_1$ such that $z \in D$. Choose any $X \in \mathcal{A}$ with $D = X(\mathbb{H})$, and define

$$T[(w, z)] = [(X^{-1}(w), X^{-1}(z))].$$

Proposition 5.3. *The formula above defines a well-defined map $T : \mathbb{M} \to \mathbb{M}$.*

Proof. First, the displayed pair lies in $\mathbb{H}_*^- \times \Lambda$. Lemma 4.9 gives $X^{-1}(z) \in \Lambda$. Since $X(\mathbb{H}) \subset \mathbb{H}$, a point $w \in \mathbb{H}_*^-$ cannot lie in $X(\mathbb{H})$. Hence $X^{-1}(w) \notin \mathbb{H}$, and therefore $X^{-1}(w) \in \mathbb{H}_*^-$.

The value is independent of the atom representing D. If $Y(\mathbb{H}) = X(\mathbb{H})$, then Lemma 4.6 gives $Y = X\gamma$ for some $\gamma \in \Gamma$. Hence

$$(Y^{-1}(w), Y^{-1}(z)) = \gamma^{-1}(X^{-1}(w), X^{-1}(z)),$$

which represents the same point of $\mathbb{M}$.

Finally, suppose $(w', z') = \gamma(w, z)$ for some $\gamma \in \Gamma$. If $D = X(\mathbb{H})$ is the atom domain containing z, then $\gamma D = (\gamma X)(\mathbb{H})$ is the atom domain containing z'. Choosing $Y = \gamma X$ gives

$$(Y^{-1}(w'), Y^{-1}(z')) = (X^{-1}(w), X^{-1}(z)).$$

Any other choice of atom representing γD gives the same class by the preceding paragraph. Thus T is well defined on the quotient. $\square$

Proposition 5.4. *The restriction of T to $\mathbb{M}_\Lambda$ is a bijection $\mathbb{M}_\Lambda \to \mathbb{M}_\Lambda$.*

Proof. First, we show that $T(\mathbb{M}_\Lambda) \subset \mathbb{M}_\Lambda$. Suppose the branch defining T is represented by $X \in \mathcal{A}$. If $w \in \Lambda^-$, then $\overline{w} \in \Lambda$ and

$$\overline{X^{-1}(w)} = X^*(\overline{w}).$$

By Lemma 4.8, $X^* \in \mathcal{A}$. Since $\overline{w} \in \Lambda$, Lemma 4.9 gives $X^*(\overline{w}) \in X^*(\Lambda) \subset \Lambda$. Therefore $X^{-1}(w) \in \Lambda^-$.

The inverse branch is determined by the first coordinate: since T applies X^{-1}, its inverse applies X, and X is recovered from the atom domain containing $\overline{w}$.

Define $S : \mathbb{M}_\Lambda \to \mathbb{M}_\Lambda$ as follows. Given $[(w,z)] \in \mathbb{M}_\Lambda$, the point $\overline{w}$ lies in Λ. Let $E \in \mathcal{D}_1$ be the unique atom domain containing $\overline{w}$. Choose $Y \in \mathcal{A}$ with $E = Y(\mathbb{H})$, put $X = Y^*$, and set

$$S[(w,z)] = [(X(w), X(z))].$$

Since $X \in \mathcal{A}$ and $z \in \Lambda$, Lemma 4.9 gives $X(z) \in X(\Lambda) \subset \Lambda$. Also, if $\overline{w} = Y(u)$ with $u \in \Lambda$, then

$$X(w) = Y^*(w) = \overline{Y^{-1}(\overline{w})} = \overline{u} \in \Lambda^-.$$

Thus the displayed pair represents a point of $\mathbb{M}_\Lambda$.

The same argument as in proposition 5.3, using Lemma 4.6, shows that S is independent of the chosen representative and of the chosen atom Y.

We now check that S is the inverse of T on $\mathbb{M}_\Lambda$. Let $[(w,z)] \in \mathbb{M}_\Lambda$, and let $X \in \mathcal{A}$ represent the atom domain containing z. Then

$$T[(w,z)] = [(X^{-1}(w), X^{-1}(z))].$$

The conjugate of the first coordinate is

$$\overline{X^{-1}(w)} = X^*(\overline{w}),$$

which lies in the atom domain $X^*(\mathbb{H})$. Therefore the inverse branch used by S is represented by $(X^*)^* = X$, and hence $S(T[(w,z)]) = [(w,z)]$.

Conversely, if S is defined using an atom X, then the second coordinate of $S[(w,z)]$ lies in $X(\Lambda)$. The corresponding branch of T is represented by X^{-1}, so $T(S[(w,z)]) = [(w,z)]$. Thus $S = T^{-1}$. $\square$

Let $\widetilde{\mu}$ be the measure on $\widehat{\mathbb{C}}^2$ induced by the density

$$d\widetilde{\mu} = \frac{d\lambda_{\mathbb{C}^2}}{|z-w|^4}. \tag{1}$$

Proposition 5.5. *For any $f \in SL(2,\mathbb{C})$, the diagonal map*

$$(w,z) \mapsto (f(w), f(z))$$

preserves $\widetilde{\mu}$.

Proof. Let

$$\Phi_f(w,z) = (f(w), f(z)).$$

We need to show that for every measurable $R \subset \mathbb{C}^2$,

$$\widetilde{\mu}(R) = \widetilde{\mu}\big(\Phi_f(R)\big).$$

Write

$$f = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C}).$$

For $(w, z) \in \mathbb{C}^2$,

$$f(w) - f(z) = \frac{(aw + b)(cz + d) - (az + b)(cw + d)}{(cw + d)(cz + d)} = \frac{w - z}{(cw + d)(cz + d)}.$$

Hence

$$|f(w) - f(z)|^4 = \frac{|w - z|^4}{|cw + d|^4 |cz + d|^4}.$$

Also,

$$f'(\zeta) = \frac{1}{(c\zeta + d)^2},$$

so the real Jacobian of f at ζ is $|f'(\zeta)|^2 = |c\zeta + d|^{-4}$. Therefore the real Jacobian of Φ_f is

$$|J_{\Phi_f}(w, z)| = \frac{1}{|cw + d|^4 |cz + d|^4}.$$

Using the change-of-variables formula, we obtain

$$\begin{aligned}
\widetilde{\mu}(\Phi_f(R)) &= \int_{R_0} \frac{|J_{\Phi_f}(w, z)|}{|f(w) - f(z)|^4} \, d\lambda_{\mathbb{C}^2}(w, z) \\
&= \int_{R_0} \frac{|cw + d|^{-4} |cz + d|^{-4}}{|w - z|^4 |cw + d|^{-4} |cz + d|^{-4}} \, d\lambda_{\mathbb{C}^2}(w, z) \\
&= \int_{R_0} \frac{d\lambda_{\mathbb{C}^2}(w, z)}{|w - z|^4} = \widetilde{\mu}(R) = \widetilde{\mu}(R).
\end{aligned}$$

This proves the claim. $\square$

On the enlarged space $\mathbb{H}^-_* \times \Lambda$, we use the measure obtained from $\widetilde{\mu}$ on $\mathbb{H}^- \times \Lambda$ and assign measure zero to the added boundary $\big(\mathbb{R} \cup \{\infty\}\big) \times \Lambda$. Since $\Gamma < SL(2, \mathbb{C})$ and Λ is Γ-invariant, proposition 5.5 shows that this measure descends to a quotient measure μ on $\mathbb{M}$.

Proposition 5.6. *(i) μ is finite and can thus be normalized to produce a probability measure. In particular $\mu(\mathbb{M}) = \pi^2/12$.*

(ii) If $[(w, z)]$ is distributed according to the normalized measure $\mu/\mu(\mathbb{M})$ on $\mathbb{M}$, then z and w are distributed on $\Gamma\backslash\mathbb{H}$ and $\Gamma\backslash\mathbb{H}^-$, respectively, with respect to normalized hyperbolic area.

Proof. Discarding the μ-null boundary part coming from $\mathbb{R} \cup \{\infty\}$, every coset in $\mathbb{M}$ has a unique representative (w, z) with $w \in \mathcal{F}^-$, up to the boundary of $\mathcal{F}^-$. Thus

$$\mu(\mathbb{M}) = \int_{\mathcal{F}^-} \int_{\Lambda} \frac{1}{|z - w|^4} \, d^2z \, d^2w.$$

Since $\mathbb{H} \setminus \Lambda$ has measure zero by proposition 4.5, this equals

$$\int_{\mathcal{F}^-} \int_{\mathbb{H}} \frac{1}{|z - w|^4} \, d^2z \, d^2w.$$

Letting $z = x + yi$ and $w = a - bi$ with $b > 0$, the inner integral becomes

$$\int_0^\infty \int_{-\infty}^\infty \frac{1}{((x-a)^2 + (y+b)^2)^2}\, dx\, dy,$$

which evaluates to

$$\int_0^\infty \frac{\pi}{2(y+b)^3}\, dy = \frac{\pi}{4b^2}. \tag{2}$$

Since $b^2 = \operatorname{Im}(w)^2$, we obtain

$$\mu(\mathbb{M}) = \frac{\pi}{4} \int_{\mathcal{F}^-} \frac{d^2 w}{(\operatorname{Im} w)^2}.$$

The integral on the right is the hyperbolic area of $\Gamma\backslash\mathbb{H}^-$. This equals the hyperbolic area of the usual modular surface $\Gamma\backslash\mathbb{H}$, namely $\pi/3$. Hence

$$\mu(\mathbb{M}) = \frac{\pi}{4} \cdot \frac{\pi}{3} = \frac{\pi^2}{12}.$$

Equation (2) also shows that the marginal distribution of w is proportional to hyperbolic area measure on $\mathcal{F}^-$. After normalization, this is normalized hyperbolic area on $\Gamma\backslash\mathbb{H}^-$. Repeating the same computation with a fundamental domain in the upper half-plane and integrating first in w shows that z is uniformly distributed according to normalized hyperbolic area on $\Gamma\backslash\mathbb{H}$. □

Proposition 5.7. *T is measure preserving with respect to μ.*

Proof. Put

$$\Sigma = \mathcal{F}^- \times \Lambda, \qquad \Sigma_\Lambda = (\mathcal{F}^- \cap \Lambda^-) \times \Lambda.$$

The set Σ is a cross-section for $\mathbb{M}$, up to the boundary and cusp null sets added to the first coordinate, and Σ_Λ has full μ-measure in Σ. The measure μ on $\mathbb{M}$ is represented on Σ by the density (1).

For each atom domain $D \in \mathcal{D}_1$, choose one atom $X_D \in \mathcal{A}$ with $D = X_D(\mathbb{H})$. On the part of Σ_Λ where $z \in D$, the quotient map T is represented by

$$(w, z) \mapsto (X_D^{-1}(w), X_D^{-1}(z)),$$

followed by reduction of the first coordinate back to $\mathcal{F}^-$. Thus, after further partitioning into the measurable sets on which a fixed $\gamma \in \Gamma$ performs this reduction, the induced map on Σ_Λ is

$$(w, z) \longmapsto (\gamma X_D^{-1}(w), \gamma X_D^{-1}(z)).$$

Each such branch is a diagonal Möbius transformation induced by an element of $SL(2, \mathbb{C})$, so proposition 5.5 shows that it preserves the density (1).

By proposition 5.4, the quotient map T is bijective on $\mathbb{M}_\Lambda$. Hence its cross-section representative maps the branch partition above bijectively onto a measurable partition of Σ_Λ, up to null sets. Since $\mathbb{M}_\Lambda$ has full μ-measure in $\mathbb{M}$, summing the branchwise equalities proves that T preserves μ. □

6 Limiting distribution

This is the section in which we prove our main result. However, we start by showing two auxiliary lemmas.

Lemma 6.1. *Let $z \in \Lambda$ and $n \geq 1$. If $B_n \in SL(2, \mathbb{Z}[i])$ satisfies $D_n(z) = B_n(\mathbb{H})$, then there is an atom $A_n \in \mathcal{A}$ such that*

$$D_{n-1}(z) = B_n A_n^{-1}(\mathbb{H}).$$

Proof. Choose $C \in SL(2, \mathbb{Z}[i])$ such that $D_{n-1}(z) = C(\mathbb{H})$; for $n = 1$, take $C = I$. Since $D_n(z)$ is the next element of the chain after $D_{n-1}(z)$, the set $C^{-1}D_n(z)$ is the first proper element in the chain of $C^{-1}(z)$. By Lemma 4.9, there is an atom $A \in \mathcal{A}$ such that

$$C^{-1}D_n(z) = A(\mathbb{H}).$$

Hence $D_n(z) = CA(\mathbb{H})$. Since also $D_n(z) = B_n(\mathbb{H})$, Lemma 4.6 gives $B_n = CA\gamma$ for some $\gamma \in \Gamma$. Then $A_n = A\gamma$ is again an atom, and

$$B_n A_n^{-1} = CA\gamma\gamma^{-1}A^{-1} = C.$$

Therefore $D_{n-1}(z) = B_n A_n^{-1}(\mathbb{H})$. □

Lemma 6.2. *Let $z \in \Lambda$. For $n \geq 0$, choose $B_n \in SL(2, \mathbb{Z}[i])$ with $D_n(z) = B_n(\mathbb{H})$, taking $B_0 = I$. Then*

$$T^n(\infty, z) = \big[(B_n^{-1}(\infty), B_n^{-1}(z))\big].$$

The right-hand side is independent of the choice of B_n.

Proof. First, the displayed pair represents a point of $\mathbb{M}$. Pulling back the tail of the chain of z under B_n gives an infinite chain through $B_n^{-1}(z)$, so $B_n^{-1}(z) \in \Lambda$. Also, $B_n^{-1}(\infty) \notin \mathbb{H}$; otherwise $\infty \in B_n(\mathbb{H}) = D_n(z)$, contrary to $D_n(z) \subset \mathbb{H}$. Hence $B_n^{-1}(\infty) \in \mathbb{H}_*^-$.

If $B'_n(\mathbb{H}) = B_n(\mathbb{H})$, then Lemma 4.6 gives $B'_n = B_n\gamma$ for some $\gamma \in \Gamma$. Therefore

$$((B'_n)^{-1}(\infty), (B'_n)^{-1}(z)) = \gamma^{-1}(B_n^{-1}(\infty), B_n^{-1}(z)),$$

so the two choices define the same point of $\mathbb{M}$.

We prove the formula by induction. The case $n = 0$ is immediate. Assume it holds for n. By Lemma 6.1, applied to $n + 1$, there is an atom $A \in \mathcal{A}$ such that

$$D_n(z) = B_{n+1}A^{-1}(\mathbb{H}).$$

By the choice-independence just proved, we may use $B_n = B_{n+1}A^{-1}$ in the induction hypothesis. Thus

$$T^n(\infty, z) = \big[(AB_{n+1}^{-1}(\infty), AB_{n+1}^{-1}(z))\big].$$

The second coordinate lies in $A(\Lambda)$, so the next application of T uses the branch represented by A^{-1}. Therefore

$$T^{n+1}(\infty, z) = \big[(B_{n+1}^{-1}(\infty), B_{n+1}^{-1}(z))\big],$$

completing the induction. □

Conjecture 1. *Let z be distributed according to normalized Euclidean area on $(0, 1) + i(0, 1)$. The points*

$$T^n(\infty, z)$$

converge in distribution to the normalized measure $\mu/\mu(\mathbb{M})$ on $\mathbb{M}$.

In order to apply this hypothesis to our question, we must be able to express γ_{n-1}/γ_n in terms of (w, z). With this in mind, for any $w \in \mathbb{H}^-$ with $\overline{w}$ in an atom domain, define $\phi(w)$ as follows: Choose an atom

$$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$$

such that $\overline{w} \in A^*(\mathbb{H})$, and set

$$\phi(w) = \frac{\operatorname{Im}\Big((a_1 w + a_2)\overline{(a_3 w + a_4)}\Big)}{\operatorname{Im}(w)}.$$

The value is independent of the choice of A. By Lemmas 4.8, 4.4, and 4.6, possible choices differ by left multiplication by an element of Γ; the displayed expression is invariant under such multiplication. On the remaining measure-zero set, define ϕ arbitrarily.

Proposition 6.3. *Let $z \in \Lambda$, and let $n \geq 1$ be such that $D_{n-1}(z)$ is a genuine disk. Choose a representative*

$$T^n(\infty, z) = [(w_n, z_n)]$$

with $w_n \in \mathbb{H}^-$. Then $\phi(w_n)$ is well-defined and

$$\frac{\gamma_{n-1}(z)}{\gamma_n(z)} = \phi(w_n).$$

Proof. By Lemma 6.2, choose $C_n \in SL(2, \mathbb{Z}[i])$ such that $D_n(z) = C_n(\mathbb{H})$ and

$$T^n(\infty, z) = \big[(C_n^{-1}(\infty), C_n^{-1}(z))\big].$$

Since (w_n, z_n) represents the same quotient point, there is a $\gamma \in \Gamma$ such that

$$(w_n, z_n) = \gamma(C_n^{-1}(\infty), C_n^{-1}(z)).$$

Replacing C_n by $B_n = C_n \gamma^{-1}$, we get

$$D_n(z) = B_n(\mathbb{H}), \qquad B_n^{-1}(\infty) = w_n, \qquad B_n^{-1}(z) = z_n.$$

By Lemma 6.1, there is an atom $A_n \in \mathcal{A}$ such that

$$D_{n-1}(z) = B_n A_n^{-1}(\mathbb{H}).$$

Put $C = B_n A_n^{-1}$. Since $C(\mathbb{H}) = D_{n-1}(z)$ is a genuine disk, $C^{-1}(\infty) \in \mathbb{H}^-$. Hence

$$A_n(w_n) = A_n B_n^{-1}(\infty) = C^{-1}(\infty) \in \mathbb{H}^-.$$

Therefore

$$\overline{w_n} = A_n^*\big(\overline{A_n(w_n)}\big) \in A_n^*(\mathbb{H}),$$

so $\phi(w_n)$ is computed using A_n.

Since $D_n(z) \subset D_{n-1}(z)$ and $D_{n-1}(z)$ is genuine, $D_n(z)$ is also genuine. Write the lower row of B_n as (u, v):

$$B_n = \begin{pmatrix} * & * \\ u & v \end{pmatrix}.$$

Then $u \neq 0$. The identity $B_n^{-1}(\infty) = w_n$ means

$$u w_n + v = 0,$$

so $v = -uw_n$. Hence

$$\gamma(B_n) = \frac{v\overline{u} - u\overline{v}}{i} = \frac{-uw_n\overline{u} - u\overline{(-uw_n)}}{i} = -2|u|^2 \mathrm{Im}(w_n).$$

Writing

$$A_n = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix},$$

the lower row of

$$A_n^{-1} = \begin{pmatrix} a_4 & -a_2 \\ -a_3 & a_1 \end{pmatrix}$$

gives the lower row of $B_n A_n^{-1}$ as

$$(ua_4 - va_3,\ -ua_2 + va_1).$$

Using $v = -uw_n$, this becomes

$$\big(u(a_3 w_n + a_4),\ -u(a_1 w_n + a_2)\big).$$

Therefore

$$\begin{aligned} \gamma(B_n A_n^{-1}) &= 2\mathrm{Im}\left(-u(a_1 w_n + a_2)\,\overline{u(a_3 w_n + a_4)}\right) \\ &= -2|u|^2 \mathrm{Im}\left((a_1 w_n + a_2)\overline{(a_3 w_n + a_4)}\right). \end{aligned}$$

By proposition 3.4,

$$\frac{\gamma_{n-1}(z)}{\gamma_n(z)} = \frac{\gamma(B_n A_n^{-1})}{\gamma(B_n)}.$$

Substituting the two expressions above and cancelling $-|u|^2$ gives

$$\frac{\gamma_{n-1}(z)}{\gamma_n(z)} = \frac{\mathrm{Im}\left((a_1 w_n + a_2)\overline{(a_3 w_n + a_4)}\right)}{\mathrm{Im}(w_n)} = \phi(w_n),$$

because $\phi(w_n)$ is computed using the atom A_n. $\square$

Let $\mu_{\mathbb{H}}$ denote hyperbolic area on $\mathbb{H}$. For any Euclidean disk $D \subset \mathbb{H}$ with centre z_0 and radius r, let

$$\alpha(D) = \frac{\mathrm{Im}(z_0)}{r}.$$

When $D = A(\mathbb{H})$ is a genuine disk with $A \in \Omega$, this equals $\alpha(A)$. Thus $\alpha(D)$ is preserved under the action of Γ.

Proposition 6.4. *Let $D \subset \mathbb{H}$ be a Euclidean disk with centre z_0, radius r, and $\alpha(D) = \mathrm{Im}(z_0)/r \geq 1$. If $\alpha(D) > 1$, then*

$$\mu_{\mathbb{H}}(D) = 2\pi \left(\frac{\alpha(D)}{\sqrt{\alpha(D)^2 - 1}} - 1 \right).$$

If $\alpha(D) = 1$, then $\mu_{\mathbb{H}}(D) = \infty$.

Proof. Write $z_0 = x_0 + iy_0$, so $\alpha(D) = y_0/r$. Hyperbolic area is invariant under horizontal translation, so assume $x_0 = 0$. If $\alpha(D) = 1$, the disk is tangent to the real line and the integral of $y^{-2}\,dx\,dy$ diverges near the tangency point.

Assume $\alpha(D) > 1$. In polar coordinates around the centre, $x = \rho\cos\theta$ and $y = y_0 + \rho\sin\theta$. Then

$$\begin{aligned}\mu_{\mathbb{H}}(D) &= \int_0^r \int_0^{2\pi} \frac{\rho}{(y_0+\rho\sin\theta)^2}\,d\theta\,d\rho \\ &= \int_0^r \frac{2\pi y_0 \rho}{(y_0^2-\rho^2)^{3/2}}\,d\rho \\ &= 2\pi y_0\left(\frac{1}{\sqrt{y_0^2-r^2}} - \frac{1}{y_0}\right) \\ &= 2\pi\left(\frac{\alpha(D)}{\sqrt{\alpha(D)^2-1}} - 1\right).\end{aligned}$$

□

Definition 6.5. For $\alpha \geq 2$, define

$$N_\alpha = \sum_{\substack{D\in\mathcal{D}_1\\ \alpha(D)=\alpha}} \frac{3\mu_{\mathbb{H}}(D\cap\mathcal{F})}{\mu_{\mathbb{H}}(D)}.$$

We also set $N_1 = 0$.

This is a natural definition, since the Apollonian strip has modular (Γ) symmetry and since hyperbolic area is preserved by the action of Γ. Equivalently, N_α is the number of disks D with $\alpha(D) = \alpha$ in the curvilinear triangle shown in figure 5.

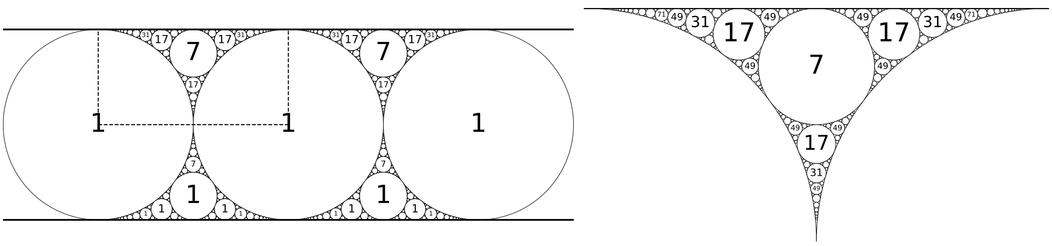


Figure 5: $\alpha(D)$ plotted for various disks in the Apollonian strip. The largest circle in the upper curvilinear triangle has $\alpha = 7$, therefore the first six values of N_α equals 0.

Theorem 6.6. *Let z be distributed according to the normalized Euclidean area on $(0,1)+i(0,1)$. Then, assuming conjecture 1,*

$$\frac{\gamma_{n-1}(z)}{\gamma_n(z)}$$

converges in distribution to the probability measure on $[0,1]$ *with density*

$$f(y) = \frac{3}{\pi} + \sum_{\alpha=2}^{\lfloor (y^2+1)/(2y) \rfloor} \frac{2N_\alpha}{\sqrt{\alpha^2-1}}, \qquad 0 < y \leq 1.$$

Proof. Put

$$R_n(z) = \frac{\gamma_{n-1}(z)}{\gamma_n(z)}.$$

By proposition 6.3, if $T^n(\infty, z) = [(w_n, z_n)]$ with $w_n \in \mathcal{F}^-$ and $D_{n-1}(z)$ is a genuine disk, then

$$R_n(z) = \phi(w_n).$$

For $z \in \Lambda$, and for n sufficiently large, $D_{n-1}(z)$ is a genuine disk; by proposition 4.5, this excludes only a null set of initial points. Therefore using conjecture 1 and proposition 5.6 imply that, for $0 < y < 1$,

$$\lim_{n\to\infty} \mathbb{P}(R_n > y) = \frac{3}{\pi} \mu_{\mathbb{H}^-}\{w \in \mathcal{F}^- : \phi(w) > y\}. \tag{3}$$

We now compute the hyperbolic area in (1). On the branch where $\overline{w} \in A^*(\mathbb{H})$, put

$$\alpha = \alpha(A^*), \quad \beta = \beta(A^*), \quad \gamma = \gamma(A^*), \quad \delta = \delta(A^*).$$

For $A \in SL(2, \mathbb{Z}[i])$, these four numbers are integers. Let $w = u + vi \in \mathbb{H}^-$. A direct expansion gives

$$\phi(w) > y \iff \gamma(u^2+v^2) + 2\beta u + 2(\alpha - y)v + \delta < 0. \tag{4}$$

First, assume $\gamma \neq 0$. The corresponding lower-half branch is the reflection of the genuine atom domain $A^*(\mathbb{H})$, and this domain has $\alpha(A^*(\mathbb{H})) = \alpha > 1$. Completing the square in (4) gives a disk E whose full hyperbolic area is

$$2\pi \left(\frac{\alpha - y}{\sqrt{\alpha^2 - 1}} - 1 \right)^+. \tag{5}$$

where $(\cdot)^+ := \max(0, \cdot)$ denotes the positive part. Indeed, this disk has

$$\alpha(E) = \frac{\alpha - y}{\sqrt{1 - 2\alpha y + y^2}},$$

so (5) follows from proposition 6.4.

Summing over the conjugated atom domains in the quotient and using the definition of N_α, the contribution from all genuine disk atom domains is

$$\sum_{\alpha \geq 2} \frac{2\pi N_\alpha}{3} \left(\frac{\alpha - y}{\sqrt{\alpha^2 - 1}} - 1 \right)^+. \tag{6}$$

It remains to consider the parabolic atom domain. It is represented by

$$A_0 = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}, \qquad A_0(z) = z + i.$$

Here

$$\alpha(A_0^*) = 1, \qquad \beta(A_0^*) = 0, \qquad \gamma(A_0^*) = 0, \qquad \delta(A_0^*) = 2,$$

so (4) becomes

$$2(1-y)v + 2 < 0,$$

or

$$v < -\frac{1}{1-y}.$$

In the quotient, this is the cusp region of hyperbolic area

$$\int_0^1 \int_{1/(1-y)}^{\infty} \frac{dt\,dx}{t^2} = 1 - y. \tag{7}$$

Combining (3), (6), and (7), we obtain

$$\lim_{n\to\infty} \mathbb{P}(R_n > y) = \frac{3}{\pi}(1-y) + \sum_{\alpha \geq 2} 2N_\alpha \left(\frac{\alpha - y}{\sqrt{\alpha^2 - 1}} - 1 \right)^+. \tag{8}$$

Finally, define f by the formula in the statement. For $0 < t \leq 1$,

$$\alpha \leq \frac{t^2+1}{2t} \iff t \leq \alpha - \sqrt{\alpha^2 - 1}.$$

Hence, for $0 < y \leq 1$,

$$\begin{aligned} \int_y^1 f(t)\,dt &= \frac{3}{\pi}(1-y) + \sum_{\alpha \geq 2} \frac{2N_\alpha}{\sqrt{\alpha^2-1}} \left(\alpha - \sqrt{\alpha^2 - 1} - y \right)^+ \\ &= \frac{3}{\pi}(1-y) + \sum_{\alpha \geq 2} 2N_\alpha \left(\frac{\alpha - y}{\sqrt{\alpha^2 - 1}} - 1 \right)^+. \end{aligned}$$

This is exactly the limiting survival function in (8). Therefore, the limiting distribution has density f, as claimed. □

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