SUPERSCARS FOR ARITHMETIC POINT SCATTERERS II

PÄR KURLBERG, STEPHEN LESTER AND LIOR ROSENZWEIG

Abstract. We consider momentum push-forwards of measures arising as quantum limits (semi-classical measures) of eigenfunctions of a point scatterer on the standard flat torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. Given any probability measure arising by placing delta masses, with equal weights, on $\mathbb{Z}^2$-lattice points on circles and projecting to the unit circle, we show that the mass of certain subsequences of eigenfunctions, in momentum space, completely localizes on that measure and are completely delocalized in position (i.e., concentration on Lagrangian states.) We also show that the mass, in momentum, can fully localize on more exotic measures, e.g. singular continous ones with support on Cantor sets. Further, we can give examples of quantum limits that are certain convex combinations of such measures, in particular showing that the set of quantum limits is richer than the ones arising only from weak limits of lattice points on circles. The proofs exploit features of the half-dimensional sieve and behavior of multiplicative functions in short intervals, enabling precise control of the location of perturbed eigenvalues.

1. Introduction

Let (M, g) be a smooth, compact Riemannian manifold with no boundary, unit mass and let Δ_g denote the Laplace-Beltrami operator. Also, let $\{\phi_\lambda\}$ be an orthonormal basis of eigenfunctions of Δ_g with eigenvalues $0 \le \lambda_1 \le \lambda_2 \le \ldots$. For an observable $f \in C^\infty(\mathbb{S}^*M)$, where $\mathbb{S}^*M$ denotes the unit co-tangent bundle of M, let $\mathrm{Op}(f)$ denote its quantization, defined as a pseudo-differential operator (cf. [9] for details.) A central problem in quantum chaos (cf. [49, Problem 3.1]) is to understand the set of possible quantum limits (sometimes called semiclassical measures) describing the distribution of mass of the eigenfunctions $\{\phi_\lambda\}$ within $\mathbb{S}^*M$, in the limit as the eigenvalue λ tends to infinity. A cornerstone result in this direction is the quantum ergodicity theorem of Shnirelman [44], Colin de Verdiére [8], and Zelditch [48] which states that if the geodesic flow on M is ergodic there exists a density one subsequence of eigenfunctions $\{\phi_{\lambda_j}\}$ such that

$$\mu_{\phi_{\lambda_j}}(f) = \langle \mathrm{Op}(f)\phi_{\lambda_j}, \phi_{\lambda_j}\rangle \to \int_{\mathbb{S}^*M} f(x)d\mu_L(x)$$

as $\lambda_j \to \infty$, where $d\mu_L$ is the normalized Liouville measure on $\mathbb{S}^*M$. (Note that any quantum limit, by Egorov's theorem, is invariant under the classical dynamics.)

While the quantum ergodicity theorem implies that the mass of almost all eigenfunctions equidistributes in $\mathbb{S}^*M$ with respect to $d\mu_L$, it does not rule out the existence of sparse subsequences along which the mass of the eigenfunctions localizes. Whether or not this happens crucially depends on the geometry of M, cf. Section 1.3.

In this article we study quantum limits of "point scatterers" on $M = \mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$. These are singular perturbations of the Laplacian on M, and were used by Šeba [39] in order to study the transition between integrability and chaos in quantum systems. The perturbation is quite weak and has essentially no effect on the classical dynamics, yet the quantum dynamics "feels" the effect of the scatterer, and an analog of the quantum ergodicity theorem is known to hold [37, 28] (namely, equidistribution holds for a full density subset of the "new" eigenfunctions.)

Date: October 10, 2020.

The model also exhibits scarring along sparse subsequences of the new eigenfunctions [26]. In particular there exist quantum limits whose momentum push-forward, which can be viewed as probability measures on the unit circle, is of the form $c\mu_{\text{sing}} + (1-c)\mu_{\text{uniform}}$, for some $c \in [1/2, 1]$. Here both μ_{uniform} and μ_{sing} are normalized to have mass one, and μ_{sing} can be taken to be a sum of delta measures giving equal mass to the four points $\pm(1,0), \pm(0,1)$. We note that μ_{uniform} is the push-forward of the Liouville measure and hence maximally delocalized, whereas μ_{sing} is maximally localized since any quantum limits in this setting must be invariant under a certain eight fold symmetry (cf. (1.5)).

Stronger localization, i.e., going strictly beyond $c = 1/2$, is particularly interesting given a number of "half delocalization" results for quantum limits for some other (strongly chaotic) systems, namely quantized cat maps and geodesic flows on manifolds with constant negative curvature. For example, in the former case Faure and Nonnenmacher showed [12] that if a quantum limit ν is decomposed as $\nu = \nu_{\text{pp}} + \nu_{\text{Liouville}} + \nu_{sc}$, with ν_{pp} denoting the pure point part and ν_{sc} denoting the singular continous part, then $\nu_{\text{Liouville}}(\mathbb{T}^2) \geq \nu_{\text{pp}}(\mathbb{T}^2)$, and thus $\nu_{\text{pp}}(\mathbb{T}^2) \leq 1/2$. (We emphasize that $\mathbb{T}^2$ is the full phase space in this setting.)

The aim of this paper is to exhibit essentially maximal localization for a quantum ergodic system, namely arithmetic toral point scatterers. In particular we construct quantum limits (in momentum) corresponding to $c = 1$ in the above decomposition; other interesting examples include singular continous measures with support, say, on Cantor sets. This can be viewed as a step towards a "measure classification" for quantum limits of quantum ergodic systems.

1.1. **Description of the model.** Let us now describe the basic properties of the point scatterer. This is discussed in further detail in [37, 38, 28, 26, 39, 41]. To describe the quantum system associated with the point scatterer, consider $-\Delta|_{D_{x_0}}$ where

$$D_{x_0} = \{f \in L^2(\mathbb{T}^2) : f(x) = 0 \text{ in some neighborhood of } x_0\}.$$

By von Neumann's theory of self-adjoint extensions (see Appendix A of [37]) there exists a one parameter family of self-adjoint extension of $-\Delta|_{D_{x_0}}$ parameterized by a phase $\varphi \in (-\pi, \pi]$. Moreover, for $\varphi \neq \pi$ the eigenvalues of these operators may be divided into two categories. The *old* eigenvalues which are eigenvalues of $-\Delta$, with multiplicity decreased by one, along with *new* eigenvalues which are solutions to the spectral equation

$$\sum_{m \geq 1} r(m) \left(\frac{1}{m-\lambda} - \frac{m}{m^2+1} \right) = \tan(\varphi/2) \sum_{m \geq 1} \frac{r(m)}{m^2+1}, \tag{1.1}$$

where

$$r(m) = \#\{(a,b) \in \mathbb{Z}^2 : a^2 + b^2 = m\}.$$

We will refer to the case when φ is fixed as $\lambda \to \infty$ the *weak coupling* quantization. In this regime work of Shigehara [41] suggests that the level spacing of the eigenvalues should have Poisson spacing statistics and this is supported by work of Rudnick and Ueberschär [38] along with Freiberg, Kurlberg and Rosenzweig [14]. In hope of exhibiting wave chaos Shigehara proposes the following *strong coupling* quantization

$$\sum_{|m-\lambda| \leq \lambda^{1/2}} r(m) \left(\frac{1}{m-\lambda} - \frac{m}{m^2+1} \right) = \frac{1}{\alpha}, \tag{1.2}$$

where $\alpha \in \mathbb{R}$ is called the physical coupling constant and reflects the strength of the scatterer. The strong coupling quantization restricts the spectral equation to the physically relevant energy levels.

Notably, this forces a re-normalization of (1.1)

$$\tan(\varphi/2)\sum_{m\geq 1}\frac{r(m)}{m^2+1}\sim -\pi\log\lambda$$

so that φ depends on λ in this case (see [47] equation (3.14)). We note that the weak coupling quantization corresponds to a fixed self adjoint extension, whereas the strong coupling quantization can be viewed as an energy dependent, albeit very slowly varying, family of self adjoint extensions.

From the spectral equation it follows that new eigenvalues interlace with integers which are representable as the sum of two integer squares. We denote these eigenvalues as follows

$$0<\lambda_0<1<\lambda_1<2<\lambda_2<4<\lambda_4<5<\lambda_5<\cdots$$

and write Λ_{new} for the set of all such eigenvalues. Also, given $n=a^2+b^2$ let n^+ denote the smallest integer greater than n which is also a sum of two squares. Let

$$\delta_n=\lambda_n-n>0, \tag{1.3}$$

(which should not be confused with the Dirac delta function). In addition given $\lambda\in\Lambda_{new}$ the associated Green's function is given by

$$G_\lambda(x)=-\frac{1}{4\pi^2}\sum_{\xi\in\mathbb{Z}^2}\frac{\exp(-i\xi\cdot x_0)}{|\xi|^2-\lambda}e^{i\xi\cdot x},\qquad g_\lambda(x)=\frac{1}{\|G_\lambda\|_2}G_\lambda(x), \tag{1.4}$$

(see equation (5.2) of [37]). Since the torus is homogeneous we may without loss of generality assume that $x_0=0$.

1.2. **Results.** Our first main result shows that along a sparse, yet relatively large, subsequence of new eigenvalues $\{\lambda_j\}$ that the mass of g_{λ_j} in momentum space localizes on measures arising from $\mathbb{Z}^2$-lattice points on circles, projected to the unit circle. To describe these measures in more detail, consider an integer $n=a^2+b^2$, with $a,b\in\mathbb{Z}$, and the following probability measure on the unit circle $S^1\subset\mathbb{C}$

$$\mu_n=\frac{1}{r(n)}\sum_{a^2+b^2=n}\delta_{(a+ib)/|a+ib|}.$$

Following Kurlberg and Wigman [30] we call a measure μ_∞ *attainable* if it is a weak limit point of the set $\{\mu_n\}_{n=a^2+b^2}$. Any such measure is invariant under rotation by $\pi/2$, as well as under reflection in the x-axis; for convenience let

$$\mathrm{Sym}_8:=\left\{\left\langle\begin{pmatrix}0&-1\\1&0\end{pmatrix},\begin{pmatrix}-1&0\\0&1\end{pmatrix}\right\rangle\right\}\subset GL_2(\mathbb{Z}) \tag{1.5}$$

denote the group generated by these transformations.

Theorem 1.1. *Let $m_0=a^2+b^2\in\mathbb{N}$ be odd*[1]*. In each of the weak and strong coupling quantizations there exists a subset of eigenvalues $\mathcal{E}_{m_0}\subset\Lambda_{new}$ with*

$$\frac{\#\{\lambda\leq X:\lambda\in\mathcal{E}_{m_0}\}}{\#\{\lambda\leq X:\lambda\in\Lambda_{new}\}}\gg\frac{1}{(\log X)^{1+o(1)}}$$

such that for any pure momentum observable $f\in C^\infty(S^1)\subset C^\infty(\mathbb{S}^*(\mathbb{T}^2))$

$$\langle \mathrm{Op}(f)g_\lambda,g_\lambda\rangle\xrightarrow[\lambda\in\mathcal{E}_{m_0}]{\lambda\to\infty}\frac{1}{r(m_0)}\sum_{a^2+b^2=m_0}f\left(\frac{a+ib}{|a+ib|}\right).$$

[1]As far as possible quantum limits go, m_0 being odd is not a restriction as any μ_n for n even can be approximated by μ_{m_0} for m_0 odd.

We note that the quantization of our observables is as explicitly given in (5.1), which follows the approach of [28].

Hence, in momentum space the mass of g_λ completely localizes on the measure μ_{m_0}. For any attainable measure μ_∞ there exists $\{m_{0,\ell}\}_\ell$ such that $\mu_{0,\ell}$ weakly converges to μ_∞. This implies the following corollary.

Corollary 1.1. *Let μ_∞ be an attainable measure. Then there exists $\{\lambda_j\}_j \subset \Lambda_{new}$ such that for any pure momentum observable $f \in C^\infty(S^1)$*

$$\langle \mathrm{Op}(f) g_{\lambda_j}, g_{\lambda_j} \rangle \xrightarrow{j\to\infty} \int_{S^1} f d\mu_\infty.$$

We note that the set of attainable measures is much smaller than the set of probabality measures on S^1 that are Sym_8-invariant, in particular the set of attainable measures is *not convex* (cf. [30, Section 3.2].) In our next result we show that in the strong coupling quantization there is a subsequence of new eigenvalues along which the entire mass of g_λ localizes on certain convex combination of two measures arising from lattice points on the circle. In particular, the set of quantum limits, in momentum space, is *strictly richer* than the set of attainable measures.

Theorem 1.2. *Let m_0, m_1 be odd integers which are each representable as a sum of two squares. Then in the strong coupling quantization there exists a subsequence of eigenvalues $\mathcal{E}_{m_0,m_1} \subset \Lambda_{new}$ such that for each $\lambda \in \mathcal{E}_{m_0,m_1}$ there is an integer ℓ_λ with $r(\ell_\lambda) \neq 0$ and $r(\ell_\lambda) \ll 1$ such that for pure momentum observables $f \in C^\infty(S^1)$*

$$\begin{aligned} \langle \mathrm{Op}(f) g_\lambda, g_\lambda \rangle =& c_\lambda \cdot \frac{1}{r(m_0)} \sum_{a^2+b^2=m_0} f\left(\frac{a+ib}{|a+ib|}\right) \\ & + (1-c_\lambda) \cdot \frac{1}{r(m_1\ell_\lambda)} \sum_{a^2+b^2=m_1\ell_\lambda} f\left(\frac{a+ib}{|a+ib|}\right) + O\left(\frac{1}{(\log\log\lambda)^{1/11}}\right), \end{aligned} \tag{1.6}$$

where

$$c_\lambda = \frac{1}{1 + r(m_0)/r(m_1\ell_\lambda)}.$$

Additionally,

$$\frac{\#\{\lambda \le X : \lambda \in \mathcal{E}_{m_0,m_1}\}}{\#\{\lambda \le X : \lambda \in \Lambda_{new}\}} \gg \frac{1}{(\log X)^{2+o(1)}}.$$

Note that since $\sum_{p|\ell_\lambda} 1 \ll 1$, the measure $\mu_{m_1\ell_\lambda}$ can be viewed as a fairly small perturbation of μ_{m_1}.

Remark 1. *By removing a further "thin" set of eigenvalues (with spectral counting function of size $O(x^{1-\epsilon})$ for $\epsilon > 0$, we can construct quantum limits that are flat in position (for details, cf.* [26, Remark 4]*), in addition to the momentum push-forward properties given in Theorems 1.1 and 1.2. In particular, we can construct quantum limits that are completely localized on the superposition of two Lagrangian states — essentially two plane waves, one in the horizontal and one in the vertical direction. This phenomena is sometimes called* **super scarring** *(cf.* [6, 26]*.)*

Further, assuming a plausible conjecture on the distribution of the prime numbers, we show that given m_0, m_1 as in Theorem 1.2 the quantum limit of $\langle \mathrm{Op}(f) g_\lambda, g_\lambda \rangle$ is a convex combination of μ_{m_0} and μ_{m_1}. From this we are able to conclude that *every* Sym_8-invariant measure arises as a quantum limit. The conjecture on the distribution of primes concerns obtaining a lower bound on the number solutions (u, v) in almost primes to the Diophantine equation

$$aX - bY = 4$$

where $v = p_1 p_2$, $u = p_3$ with p_j a prime satisfying $p_j = a_j^2 + b_j^2$ and $b_j = o(a_j)$ for $j = 1, 2, 3$. The precise formulation of this conjecture, which we call Hypothesis 1 is given in Section 5.5.

Theorem 1.3. *Assume Hypothesis 1. Let $\mu_{\infty_0}, \mu_{\infty_1}$ be attainable measures and $0 \le c \le 1$. Then in the strong coupling quantization there exists $\{\lambda_j\}_j \subset \Lambda_{new}$ such that for any $f \in C^\infty(S^1)$*

$$\langle \mathrm{Op}(f) g_{\lambda_j}, g_{\lambda_j} \rangle \xrightarrow{j \to \infty} c \int_{S^1} f d\mu_{\infty_0} + (1-c) \int_{S^1} f d\mu_{\infty_1}.$$

In particular, **all** *Sym_8-invariant probability measures on S^1 arise as quantum limits in momentum space.*

We finally remark that the proof of Theorem 1.2 easily (and unconditionally) also gives that any Sym_8-invariant probability measure μ on S^1 is a quantum limit of Greens function in the following sense: given μ, there exist a sequence of positive reals $\lambda_1' < \lambda_2' < \cdots$, disjoint from the set of unperturbed eigenvalues, so that $\lim_{i\to\infty} \langle \mathrm{Op}(f) g_{\lambda_i'}, g_{\lambda_i'} \rangle = \mu$.

1.3. **Discussion.** For integrable systems it is often straightforward to construct non-uniform quantum limits, e.g. "whispering gallery modes" for the geodesic flow in the unit ball, and for linear flows on $\mathbb{T}^2$, Lagrangian states with maximal localization (i.e., a single plane wave) are easily constructed. We note that strong localization in position for quantum limits on $\mathbb{T}^2$ was ruled out by Jakobson [20] — in position, any quantum limit is given by trigonometric polynomials whose frequencies lie on at most two circles (hence absolutely continuous with respect to Lebesgue measure.) Further, for the sphere, Jakobson and Zelditch in fact obtained a full classification — *any* flow invariant measure on $S^*(S^2)$ is a quantum limit [21].

The quantum ergodicity theorem holds in great generality as long as the key assumption of ergodic classical dynamics holds, but the existence of exceptional subsequence of nonuniform quantum limits ("scarring") is subtle. For classical systems given by the geodesic flow on compact negatively curved manifolds, the celebrated Quantum Unique Ergodicity (QUE) conjecture [36] by Rudnick and Sarnak asserts that the only possible quantum limit is the Liouville measure. Known results for QUE include Lindenstrauss' breakthrough [31] for Hecke eigenfunctions on arithmetic modular surfaces, together with Soundararajan ruling out "escape of mass" in the non-compact case [45]. On the other hand, for a generic Bunimovich stadium (with strongly chaotic classical dynamics), Hassell [16] has shown that there exists a subsequence of exceptional eigenstates where the mass localizes on sets of bouncing ball trajectories.

For quantized cat maps, again for Hecke eigenfunctions, QUE is know to hold [27]. However, unlike for arithmetic modular surfaces, where Hecke desymmetrization is believed to be unnecessary, it is essential for quantum cat maps. Namely, Faure, Nonnenmacher and de Bièvre [13] constructed, in the presence of extreme spectral multiplicities and no Hecke desymmetrization, quantum limits of the form $\nu = \frac{1}{2}\nu_{\mathrm{pp}} + \frac{1}{2}\nu_{\mathrm{Liouville}}$; in [12] this was shown to be sharp in the sense that the Liouville component always carries at least as much mass as the pure point one. (We note that, on assuming very weak bounds on spectral multiplicities, Bourgain showed [7] that scarring does not occur.) For higher dimensional analogs of quantum cat maps, Kelmer has for certain maps shown [23] "super scarring", even after Hecke desymmetrization, on invariant rational isotropic subspaces. Further, these type of scars persist on adding certain perturbations that destroy the spectral multiplicities [24]. Other models where scarring is known to exist include toral point scatterers with irrational aspect ratios [29, 22, 3] and quantum star graphs [4], though neither model is quantum ergodic [29, 4].

Classifying the set of possible quantum limits, in particular for Quantum Ergodic settings, is an interesting question. Here Anantharaman proved very strong results for geodesic flows on negatively curved manifolds [1]: any quantum limit has positive Kolmogorov-Sinai (KS) entropy with respect to the dynamics of the geodesic flow. In particular, this rules out localization on a finite number of closed geodesics (for compact arithmetic surfaces this was already known due to Rudnick and Sarnak [36].) Moreover, in the case of constant negative curvature, Anantharaman and Nonnenmacher showed [2] that the KS-entropy is at least half of the maximum possible. The measure of maximum entroy is given by the Liouville measure, and thus "eigenfunctions are at least half delocalized". Dyatlov and Jin [10] consequently showed that any quantum limit must have *full* support in $S^*(M)$, for compact hyperbolic surfaces M with constant negative curvature; together with Nonnenmacher this was recently strengthened [11] to the include the case of surfaces with variable negative curvature.

1.4. **Outline of the proofs.** Our arguments use the multiplicative structure of the integers to create an imbalance in the spectral equation (1.2) along a zero density, yet relatively large subsequence of new eigenvalues. Through exploiting this imbalance we control the location of the new eigenvalues in our subsequence and show that they lie close to integers which are sums of two squares. This greatly amplifies the amount of mass of the corresponding eigenfunctions in momentum space which lies on the terms which correspond to these integers, so much so that the contribution of the remaining terms is negligible. Consequently, the mass completely localizes on a convex combination of two measures and moreover our construction allows us to completely control the first measure.

In Section 2 we use sieve methods to produce integers $n = p_1p_2$ where p_j, $j = 1, 2$, is a prime with $p_j = a^2 + b^2 = (a + ib)(a - ib)$, $0 < b \leq a$, with $0 \leq \arctan(b/a) \leq \varepsilon$, where ε is a small parameter, such that $Q_0p_1p_2+4$ is also a sum of two squares, $Q_1|Q_0p_1p_2+4$ and $(Q_0p_1p_2+4)/Q_1$ has a bounded number of prime factors, where Q_0, Q_1 are large integers whose purpose we will describe later. In particular, we exploit special features of the half dimensional sieve using an ingenious observation of Huxley and Iwaniec [18]. Further, in order to find suitable Gaussian primes in narrow sectors we use a classical result of Hecke together with non-trivial bounds on exponential sums over finite fields to control sums of integral lattice points in narrow sectors with norms lying in arithmetic progressions to large moduli.

The subsequence of almost primes $\{n_\ell\}$ constructed as described above creates the imbalance in the spectral equation (1.2) by boosting the contribution of the terms $m = Q_0n_\ell, Q_0n_\ell+4$. The next step in our argument is to show that this imbalance typically overwhelms the contribution of the remaining terms. To do this, we first show in Section 3 that for all new eigenvalues lying outside a small exceptional set the spectral equation (1.2) can be effectively truncated to integers m with essentially $|m-\lambda| \ll (\log \lambda)^{10}$. This is done by controlling sums of $r(n)$ over short intervals and uses a second moment estimate of the Dedekind zeta-function $\zeta_{\mathbb{Q}(i)}$. In Section 4 we apply this result to new eigenvalues which lie between Q_0n_ℓ and $Q_0n_\ell + 4$ and show that for almost all such new eigenvalues the remaining terms in the spectral sum (i.e. $|m - \lambda| \ll (\log \lambda)^{10}, m \neq Q_0n_\ell, Q_0n_\ell + 4$) is relatively small, provided that we take Q_0, Q_1 sufficiently large thereby boosting the contribution of the closest two terms. This is accomplished by using bounds for sums of multiplicative functions over polynomials due to Henriot [17]. Crucially, we need good estimates for these sums in terms of the discriminant of the polynomials.

Finally, to get complete control on the first measure in Theorem 1.2 we choose Q_0 so that it is the product of a given fixed integer m_0 and large primes $p_k = a^2 + b^2$ with $0 \leq \arctan(b_k/a_k) \leq p_k^{-1/10}$ so that the probability measure on S^1 associated with Q_0n_ℓ weakly converges to the measure associated with m_0 as $\ell \to \infty$. This last construction uses work of Kubilius [25] on Gaussian primes in narrow sectors.

1.5. **Notation.** We write $f(x) \ll g(x)$ provided that $f(x) = O(g(x))$. Additionally, if for all x under consideration $|f(x)| \geq cg(x)$ we write $f(x) \gg g(x)$. If we have both $f(x) \ll g(x)$ and $f(x) \gg g(x)$ we write $f(x) \asymp g(x)$. For some additional notation related to sieves, see Section 2.1.1.

1.6. **Acknowledgments.** P.K. was partially supported by the Swedish Research Council (2016-03701). We would also like to thank Dimitris Koukouloupolos, Stéphane Nonnenmacher, Maksym Radziwiłł, Zeév Rudnick, and Steve Zelditch for very helpful discussions and suggestions.

2. Sieve estimates

Let B_0 be a sufficiently large integer, and given $\varepsilon > 0$ let

$$\begin{aligned} \mathcal{P}_\varepsilon &= \{p \geq (\log x)^{B_0} : p = a^2 + b^2 \text{ and } 0 < \arctan(b/a) \leq \varepsilon\}, \\ \mathcal{P}'_\varepsilon &= \{p \in \mathcal{P}_\varepsilon : p \leq x^{1/9}\}. \end{aligned} \tag{2.1}$$

Throughout we assume that $\varepsilon \geq 1/(\log\log x)^{1/2}$ is sufficiently small. Also given $f, g : \mathbb{N} \to \mathbb{C}$ we define the Dirichlet convolution of f and g by

$$(f * g)(n) = \sum_{ab=n} f(a)g(b).$$

Also, let $Q_0, Q_1 \leq (\log x)^{1/10}$ be odd co-prime integers whose prime factors are all $\equiv 1 \pmod 4$. Moreover we assume that $Q_0 = f_0^2 e_0 r_0^{a_0}, Q_1 = f_1^2 e_1 r_1^{a_1}$ where e_0, e_1 are square-free, $f_0, f_1 \ll 1$ and r_0, r_1 are primes congruent to $1 \pmod 4$. Throughout, the arithmetic function $b(n)$ is the indicator function of the set of integers which are representable as a sum of two squares. Also, for $\mathcal{S} \subset \mathbb{N}$ we define

$$1_{\mathcal{S}}(n) = \begin{cases} 1 & \text{if } n \in \mathcal{S}, \\ 0 & \text{otherwise.} \end{cases}$$

and let $\varphi(n) = \#\{m < n : (m, n) = 1\}$.

Proposition 2.1. *Let $\eta > 0$ be sufficiently small and let $y = x^\eta$. Suppose $y > Q_0 Q_1$. Then*

$$\sum_{\substack{n \leq x \\ Q_1 | Q_0 n + 4 \\ (\frac{Q_0 n + 4}{Q_1}, \prod_{p \leq y} p) = 1}} (1_{\mathcal{P}_\varepsilon} * 1_{\mathcal{P}'_\varepsilon})(n) b(Q_0 n + 4) \geq \frac{C\varepsilon^2 Q_0}{\eta^{1/2}\varphi(Q_0)} \cdot \frac{x \log\log x}{\varphi(Q_1)(\log x)^2},$$

for some absolute constant $C > 0$.

This proposition builds on a result of Friedlander and Iwaniec [15, Ch. 4]. The main novelty here is that we capture almost primes $n = p_1 p_2$ such that each prime factor $p = a^2 + b^2$, with $0 \leq b \leq a$, has the property that $a + ib$ lies within a certain small sector.

We also will require the following result.

Proposition 2.2. *There exists an absolute constant $C > 0$ such that*

$$\sum_{\substack{n \leq x \\ Q_1 | Q_0 n + 4}} (1_{\mathcal{P}_\varepsilon} * 1_{\mathcal{P}'_\varepsilon})(n) b(Q_0 n + 4) \geq C\varepsilon^2 \frac{x \log\log x}{\varphi(Q_1)(\log x)^{3/2}}.$$

Since Proposition 2.2 follows from a similar, yet simpler argument than the one used to prove Proposition 2.1 we will omit its proof. The rest of this section will be devoted to proving Proposition 2.1.

2.1. **The Rosser-Iwaniec Sieve.** Let us first introduce the Rosser-Iwaniec β-sieve and the classical sieve terminology. We start with a sequence of $\mathcal{A} = \{a_n\}$ of non-negative real numbers, a set of primes $\mathcal{P}$ and a parameter z. Define

$$P(z) = \prod_{\substack{p \in \mathcal{P} \\ p < z}} p.$$

Our goal is to obtain an estimate for the sieved set

$$\mathcal{S}(\mathcal{A}, \mathcal{P}, z) := \sum_{\substack{n \leq x \\ (n, P(z))=1}} a_n.$$

This will be accomplished through calculating, for square free $d \in \mathbb{N}$,

$$A_d(x) := \sum_{\substack{n \leq x \\ n \equiv 0 \pmod d}} a_n. \tag{2.2}$$

We now make the hypothesis that our estimate for $A_d(x)$ will be of the form

$$A_d(x) = g(d)X + r_d \tag{2.3}$$

where $g(d)$ is a multiplicative function with $0 \leq g(p) < 1$. The number r_d should be thought of as a remainder term, so X is an approximation to $A_1(x)$, and the function $g(d)$ can be interpreted as a density.

Let

$$V(z) = \prod_{p | P(z)} (1 - g(p)).$$

We further suppose for all $w < z$ that

$$\frac{V(w)}{V(z)} = \prod_{\substack{w \leq p < z \\ p \in \mathcal{P}}} (1 - g(p))^{-1} \leq \left(\frac{\log z}{\log w}\right)^{\kappa} \left(1 + O\left(\frac{1}{\log w}\right)\right) \tag{2.4}$$

for some $\kappa > 0$. The constant κ is referred to as the *dimension of the sieve*.

Our arguments also require sieve weights. Let $\Lambda = \{\lambda_d\}_d$, be a sequence of real numbers, where d ranges over square-free integers. The sequence Λ is referred to as an upper bound sieve provided that

$$1_{n=1} = \sum_{d|n} \mu(d) \leq \sum_{d|n} \lambda_d, \qquad \forall n \in \mathbb{N}, \tag{2.5}$$

where $1_{n=1}$ equals one if $n = 1$ and equals zero otherwise. We call Λ a lower bound sieve if

$$\sum_{d|n} \lambda_d \leq 1_{n=1}, \qquad \forall n \in \mathbb{N}. \tag{2.6}$$

For a sieve $\Lambda = \{\lambda_d\}$ we use the notation

$$(\lambda * 1)(n) = \sum_{d|n} \lambda_d. \tag{2.7}$$

(this will be used to show the existence of primes, or almost primes with desired properties.) Additionally, we say that the sieve Λ has *level* D if $\lambda_d = 0$ for $d > D$.

Given $\kappa > 0$ the β-sieve gives both an upper and lower bound for $\mathcal{S}(\mathcal{A}, \mathcal{P}, z)$ whenever $s = \log D / \log z$ is sufficiently large in terms of κ. The bounds consist of an error term, which is a sum of the remainder terms $|r_d|$ for $d \leq D$ and a main term $XV(z)F(s)$, $XV(z)f(s)$ (resp.) where F, f

are certain continuous functions with $0 \le f(s) < 1 < F(s)$. For precise definitions, motivation and context we refer the reader to [15, Chapter 11].

Theorem 2.1 (Cf. [15, Theorem 11.13])**.** *Let $D \ge z$ and write $s = \frac{\log D}{\log z}$. Then*

$$\mathcal{S}(A, \mathcal{P}, z) \le XV(z)\left(F(s) + O((\log D)^{-1/6}\right) + R(D, z)$$
$$\mathcal{S}(A, \mathcal{P}, z) \ge XV(z)\left(f(s) + O((\log D)^{-1/6}\right) - R(D, z)$$

for $s \ge \beta(\kappa) - 1$ and $s \ge \beta(\kappa)$ (resp.), where

$$R(D, z) \le \sum_{\substack{d \le D \\ d | P(z)}} |r_d|.$$

In particular, note that for $\kappa = 1/2$, it is well known that $\beta = 1$ (e.g., see [15, Ch. 14.2].) In our arguments, we will use β-sieve weights, which are as defined in [15] Sections 6.4-6.5. In particular for these weights we have $|\lambda_d| \le 1$. We will sometimes refer to the Fundamental Lemma of the Sieve, by which we mean the following result (see [15, Lemma 6.11].)

Theorem 2.2. *Let $\Lambda^{\pm} = \{\lambda_d^{\pm}\}$ be upper and lower bound (resp.) β-sieves of level D with $\beta \ge 4\kappa+1$. Also, let $s = \log D / \log z$. Then for any multiplicative function satisfying (2.4) and $s \ge \beta + 1$ we have*

$$\sum_{d | P(z)} \lambda_d^{\pm} g(d) = V(z)\left(1 + O\left(s^{-s/2}\right)\right).$$

We also require the following estimate for the convolution of two sieves (see equation (5.97) and Theorem 5.9 of [15]).

Theorem 2.3. *Let $\Lambda_1 = \{\lambda_d\}$ and $\Lambda_2 = \{\lambda_d^{'}\}$ be upper-bound sieve weights of level D_1, D_2 (resp.). Also, let g_1, g_2 be multiplicative functions satisfying (2.4) with $\kappa = 1$. Then*

$$\left|\sum_{\substack{d,e \\ (d,e)=1}} \lambda_d \lambda_e^{'} g_1(d) g_2(e)\right| \le (4e^{2\gamma} + o(1)) \prod_p (1 + h_1(p)h_2(p)) \prod_{j=1}^{2} \prod_{p < D_j} (1 - g_j(p))$$

as $\min\{D_1, D_2\} \to \infty$, where for $j = 1, 2$, $h_j(n) = g_j(n)(1 - g_j(n))^{-1}$ and γ is Euler's constant.

If in addition $g_1(p), g_2(p) \le 1/p$ so that $h_1(p)h_2(p) \ll 1/p^2$, which will be the case for us, then

$$\left|\sum_{\substack{d,e \\ (d,e)=1}} \lambda_d \lambda_e^{'} g_1(d) g_2(e)\right| \le C \prod_{p < D_1} (1 - g_1(p)) \prod_{p < D_2} (1 - g_2(p)) \tag{2.8}$$

where $C > 0$ is an absolute constant.

2.1.1. *Notation.* We will also use the notation

$$P_3(z_1, z_2) := \prod_{\substack{z_1 \le p \le z_2 \\ p \equiv 3 \pmod 4}} p, \qquad \text{and} \qquad P_3(z) := P_3(3, z).$$

Additionally, let $1(n) = 1_{\mathbb{N}}(n) = 1$ denote the identity function and let $\tau(n) = (1 * 1)(n) = \sum_{d|n} 1$. Also, define

$$\mathcal{B}(x; q, a, \varepsilon) := \sum_{\substack{n \le x \\ n \equiv a \pmod q}} (1_{\mathcal{P}_\varepsilon} * 1_{\mathcal{P}_\varepsilon^{\prime}})(n) - \frac{1}{\varphi(q)} \sum_{\substack{n \le x \\ (n,q)=1}} (1_{\mathcal{P}_\varepsilon} * 1_{\mathcal{P}_\varepsilon^{\prime}})(n). \tag{2.9}$$

Further, $\eta, \delta > 0$ will denote small, but fixed real numbers.

2.2. **Preliminary lemmas.** We begin by showing that the difference between the upper and lower bound sieves is "small".

Lemma 2.1. *Let $\Lambda^{\pm} = \{\lambda_d^{\pm}\}$ be upper and lower bound linear sieves (resp.) each of level $w = x^{\sqrt{\eta}}$ where $\eta > 0$ is sufficiently small, whose sieve weights are supported on integers d such that $d|P(y)$, where $y = x^{\eta}$ and $(d, 2Q_0 f_1 r_1) = 1$; in particular*

$$\lambda_d^{\pm} = 0 \text{ if } (d, 2Q_0 f_1 r_1) > 1. \tag{2.10}$$

Then

$$\sum_{\substack{n \le x \\ Q_1 | Q_0 n + 4}} \left((\lambda^+ * 1)\left(\frac{Q_0 n + 4}{Q_1}\right) - (\lambda^- * 1)\left(\frac{Q_0 n + 4}{Q_1}\right)\right)(1_{\mathcal{P}_\varepsilon} * 1_{\mathcal{P}'_\varepsilon})(n)$$
$$\ll \varepsilon^2 \eta^{1/(4\eta^{1/2})-1} \frac{Q_0}{\varphi(Q_0)} \frac{x \log\log x}{\varphi(Q_1)(\log x)^2} + \frac{x}{(\log x)^{10}}.$$

Proof. Switching order of summation, it follows that

$$\begin{aligned} &\sum_{\substack{n \le x \\ Q_1 | Q_0 n + 4}} \left((\lambda^+ * 1)\left(\frac{Q_0 n + 4}{Q_1}\right) - (\lambda^- * 1)\left(\frac{Q_0 n + 4}{Q_1}\right)\right)(1_{\mathcal{P}_\varepsilon} * 1_{\mathcal{P}'_\varepsilon})(n) \\ &= \sum_{\pm} \pm \sum_{\substack{d < w \\ d | P(y) \\ (d, 2Q_0 f_1 r_1) = 1}} \lambda_d^{\pm} \sum_{\substack{n \le x \\ Q_0 n + 4 \equiv 0 \ (\mathrm{mod}\ dQ_1)}} (1_{\mathcal{P}_\varepsilon} * 1_{\mathcal{P}'_\varepsilon})(n). \end{aligned} \tag{2.11}$$

The inner sum on the RHS of (2.11) equals

$$\frac{1}{\varphi(dQ_1)} \sum_{\substack{n \le x \\ (n, dQ_1) = 1}} (1_{\mathcal{P}_\varepsilon} * 1_{\mathcal{P}'_\varepsilon})(n) + \mathcal{B}(x; dQ_1, \gamma, \varepsilon) \tag{2.12}$$

where γ is the unique reduced residue $\pmod{dQ_1}$ satisfying $\gamma \cdot Q_0 \equiv -4 \pmod{dQ_1}$ and $\mathcal{B}$ is as defined in (2.9). Also,

$$\sum_{\substack{n \le x \\ (n, dQ_1) = 1}} (1_{\mathcal{P}_\varepsilon} * 1_{\mathcal{P}'_\varepsilon})(n) = \sum_{n \le x} (1_{\mathcal{P}_\varepsilon} * 1_{\mathcal{P}'_\varepsilon})(n) + O\Bigg(\sum_{\substack{p_1 p_2 \le x \\ (p_1 p_2, dQ_1) \ne 1}} 1_{\mathcal{P}_\varepsilon}(p_1) 1_{\mathcal{P}'_\varepsilon}(p_2) \Bigg). \tag{2.13}$$

Since $dQ_1 \le x^{1/9}$ (as η is small) and $p_2 \le x^{1/9}$ the contribution to the error term from $p_1 p_2 \le x$ with $p_1 | (p_1 p_2, dQ_1)$ is $\ll \sum_{p_2 \le x^{1/9}} \sum_{p_1 \le x^{1/9}} 1 \ll x^{2/9}$. Also, since $p_2 \ge (\log x)^{B_0}$

$$\sum_{\substack{p_1 p_2 \le x \\ (p_1 p_2, dQ_1) = p_2}} 1_{\mathcal{P}_\varepsilon}(p_1) 1_{\mathcal{P}'_\varepsilon}(p_2) \le \sum_{\substack{p_2 | dQ_1 \\ p_2 \ge (\log x)^{B_0}}} \sum_{p_1 \le x/p_2} 1 \ll \frac{x}{\log x} \sum_{\substack{p_2 | dQ_1 \\ p_2 \ge (\log x)^{B_0}}} \frac{1}{p_2} \ll \frac{x(\log\log x)}{(\log x)^{B_0}}. \tag{2.14}$$

Hence, using (2.12), (2.13), (2.14) along with the Fundamental Lemma of the Sieve (see Theorem 2.2 and recall $|\lambda_d| \le 1$) with $g(d) = \varphi(Q_1)/\varphi(Q_1 d)^2$, and $s = \log w/\log y = \eta^{-1/2}$ we have that

$$\begin{aligned}
&\sum_{\substack{d<w \\ d|P(y) \\ (d,2Q_0)=1}} \lambda_d^{\pm} \sum_{\substack{n\le x \\ Q_0 n+4\equiv 0 \pmod{dQ_1}}} (1_{\mathcal{P}_\varepsilon} * 1_{\mathcal{P}'_\varepsilon})(n) \\
&= \frac{1}{\varphi(Q_1)} \sum_{n\le x} (1_{\mathcal{P}_\varepsilon} * 1_{\mathcal{P}'_\varepsilon})(n) \prod_{\substack{p\le y \\ (p,2Q_0 f_1 r_1)=1}} \left(1 - \frac{\varphi(Q_1)}{\varphi(Q_1 p)}\right)(1 + O(\eta^{1/(4\eta^{1/2})})) \\
&\qquad + O\Bigg(\sum_{\substack{d<w \\ (d,2)=1}} |\mathcal{B}(x; dQ_1, \gamma, \varepsilon)| \Bigg) + O\left(\frac{x\log\log x}{(\log x)^{B_0-1}}\right).
\end{aligned} \tag{2.15}$$

Applying Theorem A.1 from the appendix, since $w = x^{\sqrt{\eta}} < x^{1/2-o(1)}$ we get that

$$\sum_{\substack{d<w \\ (d,2)=1}} |\mathcal{B}(x; dQ_1, \gamma, \varepsilon)| \ll \frac{x}{(\log x)^{10}}.$$

Using the two estimates above in (2.11) (note the main terms in (2.15) are the same for each of the sieves $\Lambda^{\pm}$ so they cancel in (2.11)) and applying (A.3) (with $q = 1$) from the appendix to estimate the sum over n, completes the proof upon noting that

$$\prod_{\substack{p\le y \\ (p,2Q_0 f_1 r_1)=1}} \left(1 - \frac{\varphi(Q_1)}{\varphi(Q_1 p)}\right) \asymp \frac{Q_0}{\varphi(Q_0)\log y} = \frac{Q_0}{\varphi(Q_0)\eta\log x}.$$

□

We next give a lower bound on the upper bound sieve, which together with Lemma 2.1 is strong enough (given suitable parameter choices) to show the existence of infinitely many integers with *exactly* two prime factors with the desired properties.

Lemma 2.2. *Let $w = x^{\sqrt{\eta}}$, $y = x^{\eta}$, and Λ^+ be as in Lemma 2.1. Let $\delta > 3\sqrt{\eta} > 0$ and $z = x^{\frac{1}{2}-\delta}$. Then there exists a constant $C_1 > 0$ such that*

$$\sum_{\substack{n\le x \\ (Q_0 n+4, P_3(y,z))=1 \\ Q_1 | Q_0 n+4}} (1_{\mathcal{P}_\varepsilon} * 1_{\mathcal{P}'_\varepsilon})(n)(\lambda^+ * 1)\left(\frac{Q_0 n+4}{Q_1}\right) \ge C_1 \frac{\varepsilon^2 \delta^{1/2}}{\eta^{1/2}} \frac{Q_0}{\varphi(Q_0)} \frac{x\log\log x}{\varphi(Q_1)(\log x)^2}.$$

Proof. Consider the sifting sequence

$$\mathcal{A} = \left\{ (1_{\mathcal{P}_\varepsilon} * 1_{\mathcal{P}'_\varepsilon})\left(\frac{m-4}{Q_0}\right)(\lambda^+ * 1)\left(\frac{m}{Q_1}\right) : Q_1 | m, Q_0 | m-4 \right\}$$

and primes $\mathcal{P} = \{p \ge y : p \equiv 3 \pmod 4\}$. Recalling (2.10), we may write

$$\begin{aligned}
X &= \sum_{\substack{e<w \\ e|P(y)}} \frac{\lambda_e^+}{\varphi(eQ_1)} \sum_{\substack{n\le x \\ (n,Q_1 e)=1}} (1_{\mathcal{P}_\varepsilon} * 1_{\mathcal{P}'_\varepsilon})(n) \\
&= \sum_{\substack{n\le x \\ (n,Q_1)=1}} (1_{\mathcal{P}_\varepsilon} * 1_{\mathcal{P}'_\varepsilon})(n) \sum_{\substack{e<w \\ e|P(y) \\ (e,2Q_0 f_1 r_1 n)=1}} \frac{\lambda_e^+}{\varphi(eQ_1)} \gg \varepsilon^2 \frac{Q_0}{\varphi(Q_0)} \frac{x\log\log x}{\varphi(Q_1)(\log y)(\log x)},
\end{aligned} \tag{2.16}$$

[2]Note that g is multiplicative on the set of square-free d with $(d, f_1 r_1) = 1$.

where the lower bound follows from the Fundamental Lemma of the Sieve (see (2.15) and take $D = w$, $z = y$ in Theorem 2.2 and note that we then have $s = \eta^{-1/2}$) along with prime number theorem for Gaussian primes in sectors to evaluate the sum over n (see (A.1), (A.3) in the Appendix).

For $d|P_3(y,z)$ note that $(d, eQ_0Q_1) = 1$ for e such that $p|e \Rightarrow p < y$, and $(1_{\mathcal{P}_\varepsilon} * 1_{\mathcal{P}'_\varepsilon})(n) = 0$ if $(d,n) \neq 1$. It follows that (cf. (2.2) and (2.3) for the definition of A_d)

$$\begin{aligned}
A_d(Q_0x+4) &= \sum_{\substack{n\le x\\ Q_1|Q_0n+4\\ Q_0n+4\equiv 0 \pmod d}} (1_{\mathcal{P}_\varepsilon} * 1_{\mathcal{P}'_\varepsilon})(n)(\lambda^+ * 1)\left(\frac{Q_0n+4}{Q_1}\right)\\
&= \sum_{\substack{e<w\\ e|P(y)}} \lambda_e^+ \sum_{\substack{n\le x\\ Q_0n+4\equiv 0 \pmod{eQ_1}\\ Q_0n+4\equiv 0 \pmod d}} (1_{\mathcal{P}_\varepsilon} * 1_{\mathcal{P}'_\varepsilon})(n)\\
&= \sum_{\substack{e<w\\ e|P(y)}} \frac{\lambda_e^+}{\varphi(deQ_1)} \sum_{\substack{n\le x\\ (n,Q_1e)=1}} (1_{\mathcal{P}_\varepsilon} * 1_{\mathcal{P}'_\varepsilon})(n) + r_d = \frac{1}{\varphi(d)}X + r_d,
\end{aligned}$$

where

$$r_d \ll \sum_{\substack{e<w\\ (e,2)=1}} |\mathcal{B}(x; deQ_1, \gamma, \varepsilon)|$$

and γ is the unique residue class $\pmod{deQ_1}$ with $Q_0\gamma \equiv -4 \pmod{eQ_1}$ and $Q_0\gamma \equiv -4 \pmod d$; also note that $(d, eQ_1) = 1$ and $\mathcal{B}$ is as in (2.9).

Hence, the half-dimensional Rosser-Iwaniec sieve, Theorem 2.1, gives for any $D \ge z$ with $s = \log D/\log z$

$$\begin{aligned}
&\sum_{\substack{n\ge 1\\ (Q_0n+4, P_3(y,z))=1\\ Q_1|Q_0n+4}} (1_{\mathcal{P}_\varepsilon} * 1_{\mathcal{P}'_\varepsilon})(n)(\lambda^+ * 1)\left(\frac{Q_0n+4}{Q_1}\right)\\
&\qquad \ge XV(z)\left(f(s) + O\left(\frac{1}{(\log D)^{1/6}}\right)\right) - \sum_{\substack{d<D\\ d|P_3(y,z)}} |r_d|
\end{aligned} \tag{2.17}$$

where

$$V(z) = \prod_{\substack{y\le p\le z\\ p\equiv 3 \pmod 4}} \left(1 - \frac{1}{p-1}\right) \gg \sqrt{\frac{\log y}{\log z}} \gg \eta^{1/2}. \tag{2.18}$$

Taking $D = z^{1+\delta}$, so $s = 1+\delta$, we have by Theorem A.1, which is proved in the appendix, that (taking $q = edQ_1$)

$$\sum_{\substack{d<D\\ d|P_3(y,z)}} |r_d| \ll \sum_{\substack{q<DQ_1w\\ (q,2)=1}} \left(\tau(q) \max_{(a,q)=1} |\mathcal{B}(x;q,a,\varepsilon)|\right) \ll \frac{x}{(\log x)^3}. \tag{2.19}$$

Here note that $DQ_1w < x^{\frac{1}{2}-\frac{\delta}{2}+\sqrt{\eta}} < x^{\frac{1}{2}-\frac{\delta}{6}}$ and the contribution of the divisor function is handled by using Cauchy-Schwarz along with the trivial bound $|\mathcal{B}(x;q,a,\varepsilon)| \ll x/q$. Also note that $f(t) \sim 2\sqrt{\frac{e^\gamma}{\pi}} \cdot \sqrt{t-1}$ as $t \to 1^+$ (see the equation after (14.3) of [15]), so $f(s) = f(1+\delta) \gg \sqrt{\delta}$. Using this along with (2.16), (2.18), and (2.19) in (2.17) completes the proof.

□

2.3. **The Proof of Proposition 2.1.** We first require a Brun-Titchmarsh type bound for primes in narrow sectors.

Lemma 2.3. *Let $Q, q \le x^{2/3-o(1)}$ be odd. Then*
$$\sum_{\substack{p=a^2+b^2\le x\\ |\arctan(b/a)|\le\varepsilon\\ qp+4=Qp_1,\, p_1 \text{ prime}}} 1 \ll \varepsilon \frac{q}{\varphi(q)}\frac{x}{\varphi(Q)(\log x)^2}.$$

Remark 2. *The point of the lemma is that it holds for large moduli $Q > x^{1/2}$. To accomplish this we use asymptotic estimates for Gaussian integers $\alpha = a+ib$ with $N(\alpha)\le x$ and $N(\alpha)\equiv a \pmod Q$ **and** $|\arg(\alpha)|\le\varepsilon$, where $N(\alpha)=\alpha\overline{\alpha}$ is the norm of α. Details are given in Appendix, cf. section A.2.*

The main step in the proof of Proposition 2.1 is the following lemma.

Lemma 2.4. *Let $z = x^{\frac{1}{2}-\delta}$ where $\delta>0$ is sufficiently small and $y = x^\eta$ with $0<\eta<1/3$. There exists a constant $C_2>0$ such that*
$$\sum_{\substack{n\le x\\ Q_1|Q_0n+4\\ (\frac{Q_0n+4}{Q_1},P(y)P_3(y,z))=1}} (1_{\mathcal{P}_\varepsilon} * 1_{\mathcal{P}'_\varepsilon})(n) = \sum_{\substack{n\le x\\ Q_1|Q_0n+4\\ (\frac{Q_0n+4}{Q_1},P(y))=1\\ p|Q_0n+4\Rightarrow p\equiv 1 \ (\mathrm{mod}\ 4)}} (1_{\mathcal{P}_\varepsilon} * 1_{\mathcal{P}'_\varepsilon})(n) + R$$
where
$$0\le R\le C_2\cdot\varepsilon^2\cdot\frac{\delta^{3/2}}{\eta^{1/2}}\frac{Q_0}{\varphi(Q_0)}\cdot\frac{x\log\log x}{\varphi(Q_1)(\log x)^2}.$$

Proof. By construction for $*1_{\mathcal{P}'_\varepsilon})(n)\ne 0$, $Q_0n+4\equiv 1 \pmod 4$ and $Q_1\equiv 1 \pmod 4$ so that $(Q_0n+4)/Q_1\equiv 1 \pmod 4$ and must have an even number of prime factors which are congruent to $3 \pmod 4$. Since $z > x^{1/4}$ the integers which contribute to R must have precisely two such prime factors. Dropping several conditions on the integers n which contribute to R, it follows that R is bounded by the number of integers $n=p_1p_2\le x$, $(1_{\mathcal{P}_\varepsilon}*1_{\mathcal{P}'_\varepsilon})(n)\ne 0$ such that $(Q_0n+4)/Q_1 = aq_1q_2$ where $b(a)=1$, $(a,P(y))=1$, $q_1\equiv q_2\equiv 3 \pmod 4$ and q_1,q_2 are primes with $z<q_1,q_2\le 2Q_0x/Q_1$ so $a\le 2Q_0x/(Q_1z^2)$. By symmetry, it suffices to consider the terms with $q_1\le q_2$. We get that
$$R\le 2\sum_{p_2\le x^{1/9}} 1_{\mathcal{P}'_\varepsilon}(p_2)\sum_{\substack{a\le\frac{2Q_0x}{Q_1z^2}\\ (a,P(y))=1}} b(a)\sum_{z<q_1\le\sqrt{\frac{2Q_0x}{aQ_1}}}\ \sum_{q_1\le q_2\le 2Q_0x/Q_1}\ \sum_{\substack{p_1\le x/p_2\\ Q_0p_1p_2+4=aq_1q_2Q_1}} 1_{\mathcal{P}_\varepsilon}(p_1). \tag{2.20}$$
Applying Lemma 2.3 with $q=Q_0p_2$ and $Q=aq_1Q_1$
$$\sum_{\substack{p_1\le x/p_2\\ Q_0p_1p_2+4=aq_1q_2Q_1}} 1_{\mathcal{P}_\varepsilon}(p_1)\ll\varepsilon\frac{Q_0}{\varphi(Q_0)}\frac{x}{\varphi(aQ_1)q_1p_2(\log x)^2}. \tag{2.21}$$
Note that $x/p_2\ge x^{8/9}$ and $Q_0p_2, aq_1Q_1\le\left(\frac{x}{p_2}\right)^{2/3-o(1)}$, for $\delta>0$ sufficiently small so the application of Lemma 2.3 is valid.

We claim that
$$\sum_{\substack{a\le\frac{2Q_0x}{Q_1z^2}\\ (a,P(y))=1}}\frac{b(a)}{\varphi(a)}\ll\sqrt{\frac{\log x/z^2}{\log y}}, \tag{2.22}$$

which we will justify below. Additionally,

$$\sum_{z<q_1\le\sqrt{\frac{2Q_0x}{aQ_1}}}\frac{1}{q_1}\sim\log\frac{\log\sqrt{\frac{2Q_0x}{aQ_1}}}{\log z}\ll\frac{\log\frac{x}{z^2}}{\log z}+\frac{\log Q_0}{\log z}\ll\frac{\log\frac{x}{z^2}}{\log z}\ll\delta. \tag{2.23}$$

Therefore, using (2.21), (2.22), and (2.23) in (2.20) we conclude that

$$\begin{aligned}R\ll&\varepsilon\cdot\frac{Q_0}{\varphi(Q_0)}\cdot\frac{x\log x/z^2}{\varphi(Q_1)(\log x)^2\log z}\sqrt{\frac{\log x/z^2}{\log y}}\sum_{p_2\le x^{1/9}}\frac{1_{\mathcal{P}'_\varepsilon}(p_2)}{p_2}\\ \ll&\varepsilon^2\cdot\frac{\delta^{3/2}}{\eta^{1/2}}\cdot\frac{Q_0}{\varphi(Q_0)}\cdot\frac{x\cdot\log\log x}{\varphi(Q_1)(\log x)^2}\end{aligned}$$

as desired.

It remains to justify (2.22). Let $F(n)$ be the completely multiplicative function defined by $F(p)=1$ if $p\ge y$ and zero otherwise. Then for all $t\ge y$, it follows from basic estimates for multiplicative functions (see (1.85) of [19]) that

$$\begin{aligned}\sum_{\substack{n\le t\\(n,P(y))=1}}b(n)\frac{n}{\varphi(n)}\le&\sum_{n\le t}b(n)\frac{n}{\varphi(n)}F(n)\\ \ll&\frac{t}{\log t}\prod_{p\le t}\left(1+\frac{b(p)F(p)}{p-1}\right)\ll\frac{t}{\sqrt{\log t\log y}}.\end{aligned}$$

For $1\le t\le y$ the sum on the LHS is empty so the bound is true in that case as well. Hence, (2.22) follows from this estimate along with partial summation. □

Proof of Proposition 2.1. Let δ be sufficiently small in terms of η, C_1 and C_2. Applying the inequality (2.6) for a lower bound sieve (also recall our notation (2.7)) along with Lemmas 2.1 and 2.2, using a lower bound sieve to take care of the condition $(\frac{Q_0n+4}{Q_1},P(y))=1$, we have that

$$\begin{aligned}\sum_{\substack{n\le x\\Q_1|Q_0n+4\\(\frac{Q_0n+4}{Q_1},P(y)P_3(y,z))=1}}(1_{\mathcal{P}_\varepsilon}*1_{\mathcal{P}'_\varepsilon})(n)\ge&\sum_{\substack{n\le x\\Q_1|Q_0n+4\\(Q_0n+4,P_3(y,z))=1}}(1_{\mathcal{P}_\varepsilon}*1_{\mathcal{P}'_\varepsilon})(n)(\lambda^-*1)\left(\frac{Q_0n+4}{Q_1}\right)\\ =&\sum_{\substack{n\le x\\Q_1|Q_0n+4\\(Q_0n+4,P_3(y,z))=1}}(1_{\mathcal{P}_\varepsilon}*1_{\mathcal{P}'_\varepsilon})(n)(\lambda^+*1)\left(\frac{Q_0n+4}{Q_1}\right)\\ &+O\left(\varepsilon^2\eta^{1/(4\eta^{1/2})-1}\frac{Q_0}{\varphi(Q_0)}\frac{x\log\log x}{\varphi(Q_1)(\log x)^2}\right)\\ \ge&C_1\frac{\varepsilon^2\delta^{1/2}}{\eta^{1/2}}\frac{Q_0}{\varphi(Q_0)}\frac{x\log\log x}{\varphi(Q_1)(\log x)^2}\left(1+O\left(\frac{\eta^{\frac{1}{4\eta^{1/2}}-\frac{1}{2}}}{\delta^{1/2}}\right)\right).\end{aligned} \tag{2.24}$$

Choosing η sufficiently small in terms of δ (which is fixed) the O-term above is $\le 1/2$ in absolute value. Therefore, by (2.24) along with Lemma 2.4 it follows that

$$\sum_{\substack{n\le x\\ Q_1|Q_0n+4\\ (\frac{Q_0n+4}{Q_1},P(y)P_3(y,z))=1\\ p|Q_0n+4\Rightarrow p\equiv 1 \pmod 4}} (1_{\mathcal{P}_\varepsilon}*1_{\mathcal{P}'_\varepsilon})(n) \ge \left(\frac{C_1}{2}\frac{\varepsilon^2\delta^{1/2}}{\eta^{1/2}} - \frac{C_2\varepsilon^2\delta^{3/2}}{\eta^{1/2}}\right)\frac{Q_0}{\varphi(Q_0)}\frac{x\log\log x}{\varphi(Q_1)(\log x)^2}.$$

The term $\left(\frac{C_1}{2}\delta^{1/2} - C_2\delta^{3/2}\right)$ is positive for δ sufficiently small in terms of C_1 and C_2. Also $b(Q_0n+4)=1$ for n such that all the prime factors of Q_0n+4 are congruent to $1 \pmod 4$. This completes the proof. ☐

3. Truncating the spectral equation

In this section we show that it is possible to achieve a very short truncation of the spectral equation which holds for almost all new eigenvalues.

Theorem 3.1. *Let $A \ge 1$. Then for $B = B(A)$ sufficiently large we have for every eigenvalue $\lambda_n \in \Lambda_{new} \cap [1,x]$ except those outside an exceptional set of size $O(x/(\log x)^A)$ that*

$$\sum_{m:|m-n|\le \frac{n}{x}(\log x)^B} \frac{r(m)}{m-\lambda_n} = \begin{cases} \pi\log\lambda_n + O(1) & \textit{in the weak coupling quantization,}\\ \frac{1}{\alpha} + O(1) & \textit{in the strong coupling quantization.}\end{cases} \tag{3.1}$$

The above theorem is proved by capturing cancellation in the spectral equation even at very small scales, for almost all new eigenvalues. This is done by showing that the average behavior of sums of $r(n)$ over even very short intervals is fairly regular.

Lemma 3.1. *Let $x \ge 3$ and $3 \le L \le x$. Also, let $h(x) = x/L$. Then*

$$\frac{1}{x}\sum_{\ell\le x}\left|\sum_{\ell\le n\le \ell+h(\ell)} r(n) - \pi h(\ell)\right|^2 \ll h(x)(\log x)^2. \tag{3.2}$$

Proof. We repeat a classical argument, which was used by Selberg [40] to study primes in short intervals. Consider

$$\zeta_{\mathbb{Q}(i)} := \frac{1}{4}\sum_{n\ge 1}\frac{r(n)}{n^s} = L(s,\chi_4)\zeta(s) \qquad \mathrm{Re}(s) > 1,$$

where $L(s,\chi_4)$ is the Dirichlet L-function attached to the non-trivial Dirichlet character $\pmod 4$, and $\zeta(s)$ denotes the Riemann zeta-function. Note $L(1,\chi_4) = \pi/4$. Applying Perron's formula, then shifting contours to $\mathrm{Re}(s) = 1/2$ (which is valid since the it is well-known that $\zeta_{\mathbb{Q}(i)}(\sigma+it) \ll t^{1-\sigma+o(1)}$, for $0 \le \sigma \le 1$) and picking up a simple pole at $s = 1$ we see that for $v, v+v/L \notin \mathbb{Z}$

$$\begin{aligned}\sum_{v\le n\le v+\frac{v}{L}} r(n) =& \frac{1}{2\pi i}\int_{(2)} 4\zeta_{\mathbb{Q}(i)}(s)\frac{(v+\frac{v}{L})^s - v^s}{s}\,ds\\ =& 4L(1,\chi_4)\cdot\frac{v}{L} + \frac{v^{1/2}}{2\pi}\int_{\mathbb{R}} 4\zeta_{\mathbb{Q}(i)}(\tfrac12+it)\frac{(1+\frac{1}{L})^{\frac12+it}-1}{\frac12+it}\cdot e^{it\log v}\,dt.\end{aligned}$$

Notice that the integral on the RHS is a Fourier transform. Writing $\nu = \log(1+\frac{1}{L})$, making a change of variables $x = e^\tau$ and then applying Plancherel's Theorem yields

$$\begin{aligned}\frac{1}{x^2}\int_1^x \Bigg(\sum_{v\le n\le v+\frac{v}{L}} r(n) - \pi\cdot\frac{v}{L}\Bigg)^2 dv \le & \int_{\mathbb{R}} \Bigg(\sum_{e^\tau\le n\le e^{\tau+\nu}} r(n) - \pi\cdot\frac{e^\tau}{L}\Bigg)^2 \frac{d\tau}{e^\tau} \\ =&\frac{8}{\pi}\int_{\mathbb{R}} |\zeta_{\mathbb{Q}(i)}(\tfrac12+it)|^2 |w_\nu(\tfrac12+it)|^2\,dt\end{aligned}$$

where $w_\nu(s) = (e^{\nu s}-1)/s \ll \min\{\nu, 1/(1+|t|)\}$ uniformly for $\frac14 \le \mathrm{Re}(s)\le 1$. To estimate the integral on the RHS we apply the well-known bound

$$\int_0^T |\zeta_{\mathbb{Q}(i)}(\tfrac12+it)|^2\,dt \ll T(\log T)^2$$

(see the introduction of [33]). Hence we see that

$$\begin{aligned}\int_{\mathbb{R}} |\zeta_{\mathbb{Q}(i)}(\tfrac12+it)|^2|w_\nu(\tfrac12+it)|^2\,dt \ll&\nu^2\int_{|t|\le 1/\nu}|\zeta_{\mathbb{Q}(i)}(\tfrac12+it)|^2\,dt + \int_{|t|\ge 1/\nu}|\zeta_{\mathbb{Q}(i)}(\tfrac12+it)|^2\frac{dt}{t^2} \\ \ll&\nu(\log 1/\nu)^2 \ll \frac{1}{L}(\log L)^2.\end{aligned}$$

Combining the estimates above we conclude that for $h = h(x) = x/L$

$$\frac{1}{x}\int_x^{2x}\Bigg(\sum_{v\le n\le v+h(v)} r(n) - \pi h(v)\Bigg)^2 dv \ll h(x)(\log x)^2. \tag{3.3}$$

We will now bound the sum over integers $\ell \le x$ on the LHS of (3.2) in terms of an integral over $1\le v\le x$. Let

$$F(v) = \sum_{v\le n\le v+h(v)} r(n) - \pi h(v)$$

and let $v_\ell \in [\ell,\ell+1]$ be a point where the minimum of $|F(v)|$ on $[\ell,\ell+1]$ is achieved. Observe that

$$F(\ell) = F(v_\ell) + O\left(r(\ell)+r(\ell^*)+1\right)$$

where $\ell^* = \lfloor \ell+1+h(\ell+1)\rfloor$. Hence,

$$\begin{aligned}\frac{1}{x}\sum_{\ell\le x}F(\ell)^2 \ll& \frac{1}{x}\sum_{\ell\le x}F(v_\ell)^2 + \frac{1}{x}\sum_{\ell\le x}(r^2(\ell)+r^2(\ell^*))+1 \\ \ll& \frac{1}{x}\int_1^x F(x)^2\,dx + \log x \ll h(x)(\log x)^2,\end{aligned}$$

where the last bound follows from (3.3).

□

Lemma 3.2. *Let $A\ge 3$ and $x, Y\ge 3$. Then for all but $\ll x/(\log x)^A$ integers $m\in[1,x]$ we have*

$$\Bigg|\sum_{Y\frac{m}{x}<k\le x^{1/2}\frac{m}{x}} \frac{r(m+k)-r(m-k)}{k}\Bigg| \le \frac{(\log x)^{3A}}{\sqrt{Y}}.$$

Proof. Let

$$R_m(t) = \sum_{1\le k\le t}(r(m+k)-r(m-k)).$$

It suffices to consider $m \in [x/(\log x)^A, x]$. Hence, by summation by parts for each integer $m \in [x/(\log x)^A, x]$ we have that

$$\sum_{Y\frac{m}{x} < k \le x^{1/2}\frac{m}{x}} \frac{r(m+k)-r(m-k)}{k} = \frac{R_m(x^{1/2}\frac{m}{x})}{x^{1/2}\frac{m}{x}} - \frac{R_m(Y\frac{m}{x})}{Y\frac{m}{x}} + \int_{Y\frac{m}{x}}^{x^{1/2}\frac{m}{x}} \frac{R_m(t)}{t^2}\,dt.$$

Using this along with Chebyshev's inequality and the elementary inequality $(|a|+|b|+|c|)^2 \le 3^2(a^2+b^2+c^2)$ it follows that

(3.4)
$$\#\left\{\frac{x}{(\log x)^A} \le m \le x : \left|\sum_{Y\frac{m}{x} < k \le x^{1/3}\frac{m}{x}} \frac{r(m+k)-r(m-k)}{k}\right| \ge \frac{(\log x)^{3A}}{\sqrt{Y}}\right\}$$
$$\le 9\frac{Y}{(\log x)^{6A}} \sum_{\frac{x}{(\log x)^A} \le m \le x} \left(\frac{R_m\left(x^{1/2}\frac{m}{x}\right)^2 (\log x)^{2A}}{x} + \frac{R_m\left(Y\frac{m}{x}\right)^2(\log x)^{2A}}{Y^2} + \left(\int_{Y\frac{m}{x}}^{x^{1/2}\frac{m}{x}} \frac{R_m(t)}{t^2}\,dt\right)^2\right).$$

In the integral we make a change of variables and apply the Cauchy-Schwarz inequality to get for each $m \in [x/(\log x)^A, x]$ that

$$\left(\int_{Y\frac{m}{x}}^{x^{1/2}\frac{m}{x}} \frac{R_m(t)}{t^2}\,dt\right)^2 \le \frac{(\log x)^{2A}}{Y}\int_Y^{x^{1/2}} \frac{1}{t^2} R_m\left(t\frac{m}{x}\right)^2\,dt. \tag{3.5}$$

Observe that

$$R_m\left(H\frac{m}{x}\right) = \sum_{m \le n \le m+\frac{m}{x}H} r(n) - \sum_{m-\frac{m}{x}H \le n \le m} r(n).$$

Hence, by Lemma 3.1 with $L = x/H$ (along with an analogue of this lemma for the second sum, which is proved in the same way) we get

$$\frac{1}{x}\sum_{m \le x} R_m\left(H\frac{m}{x}\right)^2 \ll H(\log x)^2,$$

for $1 \le H \le x/3$. Using this bound and (3.5) in (3.4) gives

$$\#\left\{\frac{x}{(\log x)^A} \le m \le x : \left|\sum_{Y\frac{m}{x} < k \le x^{1/2}\frac{m}{x}} \frac{r(m+k)-r(m-k)}{k}\right| \ge \frac{(\log x)^{3A}}{\sqrt{Y}}\right\}$$
$$\ll \frac{Y\cdot x}{(\log x)^{4A}}\left(\frac{(\log x)^2}{x^{1/2}} + \frac{(\log x)^2}{Y} + \frac{(\log x)^3}{Y}\right) \ll \frac{x}{(\log x)^{4A-3}},$$

since we may assume $Y \le x^{1/2}$ otherwise the set on the LHS above is empty. □

Before proving the main result of this section we require the following technical lemma.

Lemma 3.3. *Let u, v be sufficiently large positive real numbers such that $v^{9/10} \le u \le 2v$. Let $t > 1$ be a real number, that is not an integer which is expressible as a sum of two squares, such that $|u-t| \le v^{1/3}$. Then*

$$\sum_{m:|m-u|>v^{\frac{1}{2}}} r(m)\left(\frac{1}{m-t} - \frac{m}{m^2+1}\right) = -\pi\log t + O(1).$$

Proof. Let $A(x) = \sum_{1\le n\le x} r(n) = \pi x + E(x)$, it is well-known that (cf. [42]) that $E(x) \ll x^{\frac{1}{3}}$. Also, let $f_t(x) = \log \frac{|x-t|}{(x^2+1)^{1/2}}$, (so $f_t(x) \to 0$ as $x \to \infty$). Since $|u-t| \le v^{1/3}$, partial summation gives

$$\begin{aligned}\sum_{m:|m-u|>v^{\frac{1}{2}}} r(m)\left(\frac{1}{m-t} - \frac{m}{m^2+1}\right) &= \int_{u+v^{\frac{1}{2}}}^{\infty} f_t'(x)dA(x) + \int_{1^-}^{(u-v^{\frac{1}{2}})^-} f_t'(x)dA(x) \\ &= \pi\left(f_t(u-v^{\frac{1}{2}}) - f_t(u+v^{\frac{1}{2}}) - \log t\right) \\ &\quad + O\left(1 + \max_{\pm} \frac{u^{\frac{1}{3}}}{|u \pm v^{\frac{1}{2}} - t|}\right).\end{aligned}$$

The error is $O(1)$ since we assumed $|u-t| \le v^{1/3}$. Also,

$$f_t(u-v^{\frac{1}{2}}) - f_t(u+v^{\frac{1}{2}}) = \log \frac{|u-t-v^{\frac{1}{2}}|}{|u-t+v^{\frac{1}{2}}|} + O(1) \ll 1.$$

□

We are now ready to prove the main result of this section.

Proof of Theorem 3.1. Let $A \ge 1$. In the weak coupling quantization, it follows from the spectral equation (1.1) along with Lemma 3.3 that

$$\sum_{m:|m-n|\le \frac{n}{x}x^{1/2}} \frac{r(m)}{m-\lambda_n} = \pi \log \lambda_n + O(1) \tag{3.6}$$

for every integer $\frac{x}{(\log x)^A} \le n \le x$, which is a sum of two squares. Note that the application of Lemma 3.3 is justified since it is well-known that $\lambda_n - n \le n^+ - n \le 10n^{1/4}$ (see for instance [32] p. 43).

In the strong coupling quantization, applying Lemma 3.3 twice we get for $\frac{x}{(\log x)^A} \le n \le x$ that

$$\left|\sum_{m:|m-n|>\frac{n}{x}x^{1/2}} r(m)\left(\frac{1}{m-\lambda_n} - \frac{m}{m^2+1}\right) - \sum_{m:|m-\lambda_n|>\lambda_n^{1/2}} r(m)\left(\frac{1}{m-\lambda_n} - \frac{m}{m^2+1}\right)\right| \ll 1.$$

Hence, using this along with the spectral equation (1.2) we have

$$\begin{aligned}\sum_{|m-n|\le\frac{n}{x}x^{1/2}} r(m)\left(\frac{1}{m-\lambda_n} - \frac{m}{m^2+1}\right) &= \sum_{|m-\lambda_n|\le\lambda_n^{1/2}} r(m)\left(\frac{1}{m-\lambda_n} - \frac{m}{m^2+1}\right) + O(1) \\ &= \frac{1}{\alpha} + O(1).\end{aligned}$$

Hence, in the strong coupling quantization for each $\frac{x}{(\log x)^A} \le n \le x$

$$\sum_{m:|m-n|\le\frac{n}{x}x^{1/2}} \frac{r(m)}{m-\lambda_n} = \frac{1}{\alpha} + O(1). \tag{3.7}$$

For $\frac{x}{(\log x)^A} \le n \le x$, we now analyze the sum that appears on the LHS of both (3.6) and (3.7). Let $B \ge 1$, to be determined later and consider

$$\sum_{|m-n|\le\frac{n}{x}x^{1/2}} \frac{r(m)}{m-\lambda_n} = \sum_{|m-n|\le\frac{n}{x}(\log x)^B} \frac{r(m)}{m-\lambda_n} + \sum_{\frac{n}{x}(\log x)^B<|k|\le\frac{n}{x}x^{1/2}} \frac{r(n+k)}{k-\delta_n}, \tag{3.8}$$

where recall $\delta_n = \lambda_n - n$. Note that

$$\sum_{\substack{n\le x\\ \delta_n\ge(\log x)^{B/2}}} b(n) \le \frac{1}{(\log x)^{B/2}}\sum_{n\le x} b(n)\delta_n$$
$$\le \frac{1}{(\log x)^{B/2}}\sum_{n\le x} b(n)(n^+ - n) \ll \frac{x}{(\log x)^{B/2}}.$$

Hence, for all but $O(x/(\log x)^{B/2})$ integers $n \le x$ which are representable as a sum of two squares, $\delta_n < (\log x)^{B/2}$. For these integers, with the second sum on the RHS of (3.8) equals

$$\sum_{\frac{n}{x}(\log x)^B\le k\le \frac{n}{x}x^{1/2}} \frac{r(n+k)-r(n-k)}{k} + O\left((\log x)^{B/2}\sum_{\frac{n}{x}(\log x)^B\le |k|\le x^{1/2}} \frac{r(n+k)}{k^2}\right). \tag{3.9}$$

Since

$$\#\left\{\frac{x}{(\log x)^A}\le n\le x : (\log x)^{B/2}\sum_{\frac{n}{x}(\log x)^B\le|k|\le x^{1/2}}\frac{r(n+k)}{k^2}\ge 1\right\}$$
$$\le (\log x)^{B/2}\sum_{(\log x)^{B-A}\le |k|\le x^{1/2}}\frac{1}{k^2}\sum_{n\le x} r(n+k) \ll \frac{x}{(\log x)^{B/2-A}}$$

the O-term in (3.9) is $\ll 1$ for all but $O(x/(\log x)^{B/2-A})$ integers $\frac{x}{(\log x)^A}\le n\le x$. The first sum in (3.9) is estimated using Lemma 3.2, with $Y = (\log x)^B$; so for $B \ge 6A$ this sum is $\ll 1$ for all but at most $\ll x/(\log x)^A$ integers $n \le x$. Hence, applying the two previous estimates in (3.9) and using the resulting bound along with (3.8) in (3.6) and (3.7) completes the proof upon taking $B \ge 6A$. □

4. Estimates for new eigenvalues nearby almost primes

In this section we analyze the location of eigenvalues in Λ_{new} nearby certain integers which are almost primes. To state the result, let

$$\begin{aligned}\mathcal{N}_1 &= \{n\in\mathbb{N} : (1_{\mathcal{P}_\varepsilon} * 1_{\mathcal{P}'_\varepsilon})(n)\ne 0,\ b(Q_0n+4)=1,\ \&\ Q_1|Q_0n+4\},\\ \mathcal{N}_2 &= \left\{n\in\mathcal{N}_1 : \left(\frac{Q_0n+4}{Q_1}, P(y)\right)=1\right\},\end{aligned}\tag{4.1}$$

where $y = x^\eta$ with η as in Proposition 2.1 and $Q_0, Q_1, \varepsilon, 1_{\mathcal{P}_\varepsilon}$ and $b(\cdot)$ are as defined in the beginning of Section 2. For $j = 1,2$ let $\mathcal{N}_j(x) = \mathcal{N}_j \cap [1, x]$. In particular, for each $n \in \mathcal{N}_2(x)$, $Q_0n+4 = Q_1\ell_n$ where ℓ_n is an integer which is a sum of two squares. Moreover, since every prime divisor of ℓ_n is $\ge y = x^\eta$ so for $n \le x$, $x^{\eta\cdot\#\{p|\ell_n\}} \le \ell_n \le 2Q_0x$ and

$$\#\{p|\ell_n\} \le \frac{2}{\eta}. \tag{4.2}$$

Also, for a polynomial $R = \sum a_nX^n \in \mathbb{Z}[X]$, let $\|R\|_1 = \sum |a_n|$. Note that by Propositions 2.1 and 2.2

$$\begin{aligned}\#\mathcal{N}_1(x) &\gg \varepsilon^2\frac{1}{\varphi(Q_1)}\frac{x\log\log x}{(\log x)^{3/2}},\\ \#\mathcal{N}_2(x) &\gg \varepsilon^2\frac{Q_0}{\varphi(Q_0Q_1)}\frac{x\log\log x}{(\log x)^2}.\end{aligned}\tag{4.3}$$

Additionally by using an upper bound sieve, it is not difficult to prove that

$$
\begin{aligned}
\#\mathcal{N}_1(x) &\ll \varepsilon^2 \frac{1}{\varphi(Q_1)} \frac{x \log\log x}{(\log x)^{3/2}} \\
\#\mathcal{N}_2(x) &\ll \varepsilon^2 \frac{Q_0}{\varphi(Q_0 Q_1)} \frac{x \log\log x}{(\log x)^2}.
\end{aligned} \tag{4.4}
$$

The main result of this section is the following proposition.

Proposition 4.1. *For all $n \in \mathcal{N}_j(x)$, $j = 1, 2$, except outside an exceptional set of size*

$$
\ll \frac{\#\mathcal{N}_j(x)}{\varepsilon^2 (\log\log x)^{1-o(1)}}
$$

we have for $m = Q_0 n$ that $m^+ = m + 4$ and

$$
\begin{aligned}
&\frac{r(m)}{m - \lambda_m} + \frac{r(m^+)}{m^+ - \lambda_m} \\
&\qquad = \begin{cases} \pi \log \lambda_m + O\left((\log\log x)^5\right) & \textit{in the weak coupling quantization,} \\ O\left((\log\log x)^5\right) & \textit{in the strong coupling quantization.} \end{cases}
\end{aligned}
$$

We also require a sieve estimate for averages of correlations of multiplicative functions. The following result is due to Henriot [17], which builds on the work of Nair and Tenenbaum [34]. See Corollary 1 of [17] and the subsequent remark therein. Recall that $\tau(n) = \sum_{d|n} 1$ denotes the divisor function.

Lemma 4.1. *Let $R_1(X), \ldots, R_k(X) \in \mathbb{Z}[X]$ be irreducible, pairwise co-prime polynomials, for which each polynomial R_j does not have a fixed prime divisor. Let D be the discriminant of $R = R_1 \cdots R_k$ and $\varrho_{R_j}(n) = \#\{a \pmod n : R_j(a) \equiv 0 \pmod n\}$. Then there exist $C, c_0 > 0$ such that for any non-negative multiplicative functions F_j, $j = 1, \ldots, k$ with $F_j(n) \le \tau(n)$, we have for $x \ge c_0 \|R\|_1^{1/10}$ and some $A \ge 1$ that*

$$
\sum_{n \le x} \prod_{j=1}^{k} F_j(|R_j(n)|) \ll \Delta_D \, x \prod_{p \le x} \left(1 - \frac{\varrho_R(p)}{p}\right) \prod_{j=1}^{k} \left(\sum_{n \le x} \frac{F_j(n) \varrho_{R_j}(n)}{n} \right)
$$

where

$$
\Delta_D := \prod_{p|D} \left(1 + \frac{1}{p}\right)^C,
$$

and the implicit constant, C and c_0 depend at most on the degree of R.

We first start with a technical lemma.

Lemma 4.2. *Let f be a non-negative multiplicative function with $f(n) \le \tau(n)$ and $f(mn) \le \max\{1, f(n)\} f(m)$ for $m \in \mathbb{N}$ and n such that $b(n) = 1$. Then for $1 \le |h| \le x^{1/30}$, with $h \ne 4$ and $j = 1, 2$, we have*

$$
\sum_{n \in \mathcal{N}_j(x)} f(Q_0 n + h) \ll \frac{1}{\varepsilon^2} \cdot g(h) \prod_{p | Q_0 Q_1} \left(1 + \frac{1}{p}\right)^C \prod_{p \le x} \left(1 + \frac{f(p) - 1}{p}\right) \#\mathcal{N}_j(x) \tag{4.5}
$$

where $C > 0$ is an absolute constant and

$$
g(h) = \tau(|h|)\tau(|h-4|) \prod_{p|h} \left(1 + \frac{1}{p}\right)^C \prod_{p|h-4} \left(1 + \frac{1}{p}\right)^C.
$$

Additionally (for $h = 4$) there exists $C > 0$ such that

$$\sum_{n \in \mathcal{N}_1(x)} f(Q_0 n + 4) \ll \frac{1}{\varepsilon^2} \cdot f(Q_1) \prod_{p | Q_0 Q_1} \left(1 + \frac{1}{p}\right)^C \prod_{\substack{p \le x \\ p \equiv 1 \ (\mathrm{mod}\ 4)}} \left(1 + \frac{f(p) - 1}{p}\right) \# \mathcal{N}_1(x).$$

Remark 3. *When applying this lemma we will take $f(n) = \frac{1}{4} \cdot r(n), b(n)$ or $2^{-\omega_1(n)}$ where $\omega_1(n) = \#\{p | n : p \equiv 1 \pmod 4\}$. The hypotheses of the lemma are satisfied for each of these choices.*

Proof. Let $T_j = 2$ if $j = 1$ and $T_j = y$ if $j = 2$. Dropping several of the conditions on $n \in \mathcal{N}_j$ we get that (here $q < p$ denote primes)

$$\sum_{n \in \mathcal{N}_j(x)} f(Q_0 n + h) \le 2 \sum_{\substack{q \le \sqrt{x} \\ q \equiv 1 \ (\mathrm{mod}\ 4)}} \sum_{\substack{p \le x/q \\ Q_1 | Q_0 pq + 4 \\ (\frac{Q_0 pq + 4}{Q_1}, P(T_j)) = 1}} b(Q_0 qp + 4) f(Q_0 qp + h). \tag{4.6}$$

Let $K = Q_0 q$ and $Y = x/q$. Note that the sum above is empty unless $(K, Q_1) = 1$. Since $(K, Q_1) = 1$ there exist integers $\overline{K}, \overline{Q}_1$ with $1 \le |\overline{K}| < Q_1$ and $1 \le |\overline{Q_1}| < K$ such that $K\overline{K} - Q_1 \overline{Q}_1 = 1$. Also, for $Z \ge 1$ let F_Z be the totally multiplicative function given by $F_Z(p) = 1$ if $p \ge Z$ and zero otherwise. The inner sum on the RHS of (4.6) is bounded by

$$\begin{aligned} &\ll \sum_{\substack{n \le Y \\ Q_1 | Kn + 4}} F_{\sqrt{Y}}(n) F_{T_j}\left(\frac{Kn + 4}{Q_1}\right) b(Kn + 4) f(Kn + h) + Y^{1/2 + o(1)} \\ &= \sum_{m \le \frac{Y - 4\overline{K}}{Q_1}} F_{\sqrt{Y}}(Q_1 m - 4\overline{K}) F_{T_j}\left(Km - 4\overline{Q_1}\right) b(KQ_1 m - 4Q_1 \overline{Q_1}) f(KQ_1 m + h - 4K\overline{K}) \\ &\quad + O(Y^{1/2 + o(1)}). \end{aligned} \tag{4.7}$$

First note $b(KQ_1 n - 4Q_1\overline{Q_1}) = b(Kn - 4\overline{Q_1})$. Let $d = (KQ_1, h - 4K\overline{K})$ and suppose that $h \ne 4$. We have

$$f(KQ_1 m + h - 4K\overline{K}) \le \max\{1, f(d)\} f\left(\frac{KQ_1}{d} m + \frac{h - 4K\overline{K}}{d}\right).$$

Let $R_1(X) = Q_1 X - 4\overline{K}$, $R_2(X) = KX - 4\overline{Q}_1$, $R_3(X) = \frac{KQ_1}{d} X + \frac{h - 4K\overline{K}}{d}$ and D denote the discriminant of $R = R_1 R_2 R_3$. The polynomials R_1, R_2, R_3 and multiplicative functions $F_1 = F_{\sqrt{Y}}$, $F_2 = F_{T_j} \cdot b$ and $F_3 = f$ satisfy the assumptions of Lemma 4.1. Also for $(p, KQ_1) = 1$ we have $\varrho_R(p) = 3$ and $\varrho_{R_j}(p^k) = 1$ for each $j = 1, 2, 3$ and $k \ge 1$, which follows from Hensel's lemma. Hence, the sum in (4.7) is bounded by

$$\begin{aligned} &\ll \max\{1, f(d)\} \Delta_D \frac{Y}{Q_1} \prod_{p \le Y} \left(1 + \frac{F_{\sqrt{Y}}(p) + F_{T_j}(p) b(p) + f(p) - 3}{p}\right) \prod_{p | KQ_1} \left(1 + \frac{1}{p}\right)^C \\ &\ll \max\{1, f(d)\} \Delta_D \prod_{p | KQ_1} \left(1 + \frac{1}{p}\right)^C \frac{Y}{Q_1 (\log Y)^{3/2} (\log T_j)^{1/2}} \prod_{p \le Y} \left(1 + \frac{f(p) - 1}{p}\right). \end{aligned}$$

Write $d = p_1^{a_1} \cdots p_\ell^{a_\ell}$. For each $j = 1, \ldots, \ell$ we have $p_j^{a_j} | h$ or $p_j^{a_j} | h - 4$ (depending on whether $p_j^{a_j} | K$ or $p_j^{a_j} | Q_1$, respectively); so $f(d) \ll \tau(|h|)\tau(|h - 4|)$. Note the discriminant of R equals

$D = 16\frac{K^2Q_1^2}{d^4}h^2(h-4)^2$ so that

$$\max\{1, f(d)\}\Delta_D \ll g(h) \prod_{p|Q_1K} \left(1 + \frac{1}{p}\right)^C.$$

Also since $Y = x/q \geq \sqrt{x}$, $\prod_{p\leq Y}\left(1 + \frac{f(p)-1}{p}\right) \ll \prod_{p\leq x}\left(1 + \frac{f(p)-1}{p}\right)$. Hence, applying the estimates above in (4.6), summing over q and using (4.3) gives the claimed bound for $h \neq 4$.

For $h = 4$ we argue similarly, only now in order to estimate (4.7) we use Lemma 4.1 with R_1, R_2 as before, $R = R_1R_2$ (so the discriminant is $D = 16$) and $F_1 = F_{\sqrt{Y}}$, $F_2 = b \cdot f$. Also noting that here $d = Q_1$ we conclude that (4.7) is bounded by

$$\begin{aligned}
&\ll f(Q_1) \prod_{p|Q_1K}\left(1 + \frac{1}{p}\right)^C \frac{Y}{Q_1(\log Y)^2} \prod_{p\leq x}\left(1 + \frac{b(p)f(p)}{p}\right)\\
&\ll f(Q_1) \prod_{p|Q_1K}\left(1 + \frac{1}{p}\right)^C \frac{Y}{Q_1(\log x)^{3/2}} \prod_{\substack{p\leq x\\ p\equiv 1 \ (\mathrm{mod}\ 4)}}\left(1 + \frac{f(p)-1}{p}\right).
\end{aligned}$$

Hence, the claim follows in the same way as before. $\square$

Lemma 4.3. *Let* $(\log\log x)^4 \leq U \leq \frac{1}{10}(\log x)^{1/2}$. *There exists* $C > 0$ *such that for all* $n \in \mathcal{N}_j(x)$, $j = 1, 2$, *outside a set of size*

$$\ll \frac{1}{\varepsilon^2} \cdot \#\mathcal{N}_j(x) \prod_{p|Q_1Q_0}\left(1 + \frac{1}{p}\right)^C \frac{(\log\log x)^4}{U}$$

the following hold:

$$\sum_{\substack{1\leq|k|\leq\frac{1}{U}(\log x)^{1/2}\\ k\neq 4}} b(Q_0n + k) = 0, \tag{4.8}$$

$$\sum_{\substack{1\leq|k|\leq\frac{n}{x}(\log x)^{B}\\ k\neq 4}} \frac{r(Q_0n + k)}{|k|} \leq U, \tag{4.9}$$

and

$$\sum_{|k|\geq U} \frac{r(Q_0n + k)}{k^2} \leq \frac{1}{\log\log x}. \tag{4.10}$$

Proof. We first establish (4.8). By Chebyshev's inequality

$$\#\Big\{n \in \mathcal{N}_j(x) : \sum_{\substack{1\leq|k|\leq\frac{1}{U}(\log x)^{1/2}\\ k\neq 4}} b(Q_0n + k) \geq 1\Big\} \leq \sum_{\substack{1\leq|k|\leq\frac{1}{U}(\log x)^{1/2}\\ k\neq 4}} \sum_{n\in\mathcal{N}_j(x)} b(Q_0n + k). \tag{4.11}$$

Applying Lemma 4.2 to the inner sum and noting that

$$\prod_{p\leq x}\left(1 + \frac{b(p)-1}{p}\right) \ll \frac{1}{\sqrt{\log x}}$$

we get that the LHS of (4.11) is bounded by

$$
\begin{aligned}
&\ll \prod_{p|Q_1Q_0}\left(1+\frac{1}{p}\right)^C \frac{\#\mathcal{N}_j(x)}{\varepsilon^2\sqrt{\log x}} \sum_{\substack{1\le|k|\le\frac{1}{U}(\log x)^{1/2}\\ k\neq 4}} g(k)\\
&\ll \prod_{p|Q_1Q_0}\left(1+\frac{1}{p}\right)^C \frac{\#\mathcal{N}_j(x)}{\varepsilon^2}\frac{(\log\log x)^2}{U},
\end{aligned}
\tag{4.12}
$$

where the second step follows upon using Lemma 4.1.

To prove (4.9), we argue similarly and apply Lemmas 4.1 and 4.2 to get

$$
\begin{aligned}
\#\Bigg\{n\in\mathcal{N}_j(x): &\sum_{\substack{1\le|k|\le\frac{n}{x}(\log x)^B\\ k\neq 4}} \frac{r(Q_0n+k)}{|k|}>U\Bigg\}\\
&\le \frac{1}{U}\sum_{\substack{1\le|k|\le(\log x)^B\\ k\neq 4}}\frac{1}{|k|}\sum_{n\in\mathcal{N}_j(x)} r(Q_0n+k)\\
&\ll \frac{\#\mathcal{N}_j(x)}{\varepsilon^2U}\prod_{p|Q_0Q_1}\left(1+\frac{1}{p}\right)^C\sum_{\substack{1\le|k|\le(\log x)^B\\ k\neq 4}}\frac{g(k)}{|k|}\\
&\ll \frac{\#\mathcal{N}_j(x)}{\varepsilon^2U}\prod_{p|Q_0Q_1}\left(1+\frac{1}{p}\right)^C(\log\log x)^3.
\end{aligned}
$$

We will omit the proof of (4.10) since it follows similarly. ◻

For almost all $n\in\mathcal{N}_1(x)$ it is possible to show that $r(Q_0n+4)\asymp(\log n)^{\log 2/2\pm o(1)}$, however since we do not actually need this estimate we will record the weaker estimate below, which suffices for our purposes and is simpler to prove.

Lemma 4.4. *Let $\nu>0$ be sufficiently small. There exists $C>0$ such that for all $n\in\mathcal{N}_1(x)$ outside a set of size*

$$
\ll \frac{1}{\varepsilon^2}\#\mathcal{N}_1(x)\frac{(\log\log x)^C}{(\log x)^\nu}
$$

the following holds

$$
(\log x)^{1/4-\nu}\le r(Q_0n+4)\le(\log x)^{1/2+\nu}. \tag{4.13}
$$

Proof. We will only prove the lower bound stated in (4.13). Let $\omega_1(n)=\sum_{\substack{p|n\\ p\equiv 1 \pmod 4}}1$. For n which is a sum of two squares $r(n)\ge 2^{\omega_1(n)}$. Using this with Chebyshev's inequality and Lemma 4.2 the number of $n\in\mathcal{N}_1(x)$ which $r(Q_0n+4)<(\log x)^{1/4-\nu}$ is bounded by

$$
\begin{aligned}
(\log x)^{1/4-\nu}\sum_{n\in\mathcal{N}_1(x)}2^{-\omega_1(Q_0n+4)} &\ll (\log x)^{1/4-\nu}\cdot\frac{(\log\log x)^C}{\sqrt{\log x}}\prod_{\substack{p\le x\\ p\equiv 1 \pmod 4}}\left(1+\frac{1}{2p}\right)\cdot\frac{1}{\varepsilon^2}\#\mathcal{N}_1(x)\\
&\ll \frac{1}{\varepsilon^2}\#\mathcal{N}_1(x)\frac{(\log\log x)^C}{(\log x)^\nu}
\end{aligned}
$$

using Lemma 4.2. ◻

Proof of Proposition 4.1. By Theorem 3.1 we get for all but $O(x/(\log x)^A)$ new eigenvalues $\lambda_\ell \le x$ that

$$\sum_{|m-\ell|\le \frac{\ell}{x}(\log x)^B} \frac{r(m)}{m-\lambda_\ell} = \begin{cases} \pi \log \lambda_\ell + O(1) & \text{in the weak coupling quantization,} \\ \frac{1}{\alpha} + O(1) & \text{in the strong coupling quantization.} \end{cases}$$

We now consider integers $\ell = Q_0 n$ with $n \in \mathcal{N}_j(x)$, $j = 1, 2$ such that the above holds. Using Lemma 4.3, in particular (4.8) and (4.9) with $U = (\log\log x)^5$ it follows that for all but $O(\#\mathcal{N}_j/(\varepsilon^2(\log\log x)^{1-o(1)}))$ of these integers $n \in \mathcal{N}_j(x)$, $j = 1, 2$, with $\ell = Q_0 n$ that $\ell^+ = \ell + 4$ and

$$\sum_{|m-\ell|\le \frac{\ell}{x}(\log x)^B} \frac{r(m)}{m-\lambda_\ell} = \frac{r(\ell)}{\ell-\lambda_\ell} + \frac{r(\ell^+)}{\ell^+-\lambda_\ell} + O\left((\log\log x)^5\right).$$

Combining the two estimates above completes the proof. □

5. Proofs of the main theorems

5.1. **Quantization of Observables.** On the unit cotangent bundle $\mathbb{S}^*M \cong \mathbb{T}^2 \times S^1$, a smooth function $f \in C^\infty(S^1)$ has the Fourier expansion

$$f(x,\phi) = \sum_{\zeta\in\mathbb{Z}^2, k\in\mathbb{Z}} \widehat{f}(\zeta,k) e^{i\langle x,\zeta\rangle + ik\phi}.$$

Following Kurlberg and Ueberschär [28], we quantize our observables as follows. For $g \in L^2(S^1)$ let

$$(\mathrm{Op}(f)g)(x) = \sum_{\xi\in\mathbb{Z}^2\setminus 0}\sum_{\zeta\in\mathbb{Z}^2,k\in\mathbb{Z}} \widehat{f}(\zeta,k)e^{ik\arg\xi}\widehat{g}(\xi)e^{i\langle\zeta+\xi,x\rangle} + \sum_{\zeta\in\mathbb{Z}^2,k\in\mathbb{Z}} \widehat{f}(\zeta,k)\widehat{g}(0)e^{i\langle\zeta,x\rangle}. \tag{5.1}$$

Hence, for pure momentum observables $f : S^1 \to \mathbb{R}$ one has

$$(\mathrm{Op}(f)g)(x) = \sum_{\xi\in\mathbb{Z}^2} f\left(\frac{\xi}{|\xi|}\right)\widehat{g}(\xi)e^{i\langle\xi,x\rangle} \tag{5.2}$$

and for $\xi = 0$, $f(\frac{\xi}{|\xi|})$ is defined to be $\int_{S^1} f(\theta)\,\frac{d\theta}{2\pi}$.

Let g_λ be as given in (1.4). Then for f a pure momentum observable it follows from (1.4) and (5.2) that

$$\begin{aligned} \langle \mathrm{Op}(f)g_\lambda, g_\lambda\rangle &= \frac{1}{16\pi^4}\cdot\frac{1}{\|G_\lambda\|_2^2}\sum_{n\ge 0}\frac{1}{(n-\lambda)^2}\sum_{a^2+b^2=n} f\left(\frac{a+ib}{|a+ib|}\right) \\ &= \frac{1}{\sum_{n\ge0}\frac{r(n)}{(n-\lambda)^2}}\sum_{n\ge 0}\frac{1}{(n-\lambda)^2}\sum_{a^2+b^2=n} f\left(\frac{a+ib}{|a+ib|}\right). \end{aligned} \tag{5.3}$$

5.2. **Measures associated to sequences of almost primes in narrow sectors.** Let $\mathcal{N}_1, \mathcal{N}_2$ be as in (4.1). Before proceeding to the main result of this section we will specify our choice of Q_0, Q_1. Consider the set of primes

$$\mathcal{S} = \{p : p = a^2+b^2, 0 \le b \le a \text{ and } 0 < \arctan(b/a) \le p^{-1/10}\} \tag{5.4}$$

and let q_j be the jth element of $\mathcal{S}$. It follows from work of Kubilius [25] that

$$\#\{p \le x : p \in \mathcal{S}\} \asymp \frac{x^{9/10}}{\log x},$$

so $q_j \asymp (j\log j)^{10/9}$. Let $T = \lfloor \log\log x\rfloor$, $H = \lfloor 100\log\log\log x\rfloor$ and

$$Q_0' = \prod_{j=T}^{T+H-1} q_j, \qquad Q_1' = \prod_{j=T+H}^{T+2H-1} q_j. \tag{5.5}$$

Also, let $r_0, r_1 \in \mathcal{S}$ with $\frac{1}{4}\log\log x \le r_0, r_1 \le \frac{1}{2}\log\log x$ and $a_0, a_1 \in \mathbb{Z}$ with $0 \le a_0, a_1 \le \log\log\log x$. Let m_0, m_1 be integers, which are fixed (in terms of x), whose prime factors are all congruent to 1 (mod 4). Write $(m_0, m_1) = p_1^{e_1}\cdots p_s^{e_s}$ and let $g' = \widetilde{p}_1^{e_1}\cdots\widetilde{p}_s^{e_s}$ where $\frac{1}{2}\log\log x < \widetilde{p}_j < \log\log x$, $\widetilde{p}_j = c_j^2+d_j^2$ with $0 \le c_j \le d_j$ and $\arctan(c_j/d_j) = \arctan(b_j/a_j) + O(1/(\log\log x)^{1/10})$ where $a_j^2+b_j^2 = p_j$ with $0 \le b_j \le a_j$, for each $j = 1,\dots,s$. We now take

$$Q_0 = Q_0' m_0 r_0^{a_0}, \qquad Q_1 = Q_1' \frac{m_1}{(m_0,m_1)} r_1^{a_1} g'. \tag{5.6}$$

Note that $(Q_0, Q_1) = 1$ and that $Q_0, Q_1 \ll \exp(200(\log\log\log x)^2) \le (\log x)^{1/10}$ so that this choice of Q_0, Q_1 is consistent with our prior assumption. For $j = 1, 2$ let

$$\mathcal{M}_j(x) = \{m \le x : m = Q_0 n \text{ and } n \in \mathcal{N}_j\}. \tag{5.7}$$

By (4.3) and (4.4),

$$\#\mathcal{M}_1(x) \asymp \varepsilon^2 \frac{1}{\varphi(Q_1)} \frac{x\log\log x}{Q_0(\log x)^{3/2}} \tag{5.8}$$

and

$$\#\mathcal{M}_2(x) \asymp \varepsilon^2 \frac{1}{\varphi(Q_0Q_1)} \frac{x\log\log x}{(\log x)^{2}}. \tag{5.9}$$

We also now assume that

$$\varepsilon = (\log\log x)^{-1/4}$$

Lemma 5.1. *Let Q_0, Q_1 be as in (5.6) and $\varepsilon, \eta > 0$ be as in Proposition 2.1. Let $m \in \mathcal{M}_j(x)$, $j = 1, 2$ where $\mathcal{M}_j(x)$ is defined as in (5.7). Then for $f \in C^1(S^1)$ with $|f'| \ll 1$*

$$\frac{1}{r(m)} \sum_{a^2+b^2=m} f\left(\frac{a+ib}{|a+ib|}\right) = \frac{1}{r(m_0)} \sum_{a^2+b^2=m_0} f\left(\frac{a+ib}{|a+ib|}\right) + O(\varepsilon). \tag{5.10}$$

Under the same hypotheses, we have for $m = Q_0 n \in \mathcal{N}_2(x)$ that there exists an integer ℓ_n which is a sum of two squares with $\#\{p|\ell_n\} \le 2/\eta$ such that

$$\frac{1}{r(m^+)} \sum_{a^2+b^2=m^+} f\left(\frac{a+ib}{|a+ib|}\right) = \frac{1}{r(m_1\ell_n)} \sum_{a^2+b^2=m_1\ell_n} f\left(\frac{a+ib}{|a+ib|}\right) + O\left(\frac{1}{(\log\log x)^{1/11}}\right). \tag{5.11}$$

Proof. First note that for a unit, u of $\mathbb{Z}[i]$ i.e. $u \in \{\pm 1, \pm i\}$, that for any $n \in \mathbb{N}$

$$\sum_{a^2+b^2=n} f\left(\frac{u(a+ib)}{|a+ib|}\right) = \sum_{a^2+b^2=n} f\left(\frac{a+ib}{|a+ib|}\right). \tag{5.12}$$

For $m \in \mathcal{M}_j(x)$ with $j = 1$ or $j = 2$ write $m = Q_0' m_0 r_0^{a_0} n$ where $n \in \mathcal{N}_j(x)$. The factorizations of the ideals $(m) = ((a+ib)(a-ib))$ in $\mathbb{Z}[i]$ are in one-to-one correspondence with factorizations $(Q_0') = ((c+id)(c-id))$, $(m_0) = ((e+if)(e-if))$, $(r_0^{a_0}) = ((g+ih)(g-ih))$ and $(n) = ((k+il)(k-il))$, since Q_0', m_0, n are pairwise co-prime. Hence, it follows from this and (5.12) that

$$\frac{1}{r(m)} \sum_{a^2+b^2=m} f\left(\frac{a+ib}{|a+ib|}\right) = \frac{1}{r(Q_0')r(m_0)r(r_0^{a_0})r(n)} \sum_{\substack{\alpha\in\mathbb{Z}[i]\\ \alpha\overline{\alpha}=Q_0'}} \sum_{\substack{\beta\in\mathbb{Z}[i]\\ \beta\overline{\beta}=m_0}} \sum_{\substack{\gamma\in\mathbb{Z}[i]\\ \gamma\overline{\gamma}=r_0^{a_0}}} \sum_{\substack{\delta\in\mathbb{Z}[i]\\ \delta\overline{\delta}=n}} f\left(\frac{\alpha\beta\gamma\delta}{|\alpha\beta\gamma\delta|}\right). \tag{5.13}$$

Let $\mathcal{S}$ be as in (5.4) and write the jth element of $\mathcal{S}$ as $q_j = a_j^2+b_j^2$, with $0 \le b_j \le a_j$. By construction, for $\alpha \in \mathbb{Z}[i]$ with $\alpha\overline{\alpha} = Q_0'$ we can write $\alpha = u\prod_{j\in J}(a_j+\epsilon_j ib_j)$ where $J = \{T, T+1, \dots, T+H_1-1\}$, $\epsilon_j \in \{\pm 1\}$ and u is a unit. It follows that

$$\begin{aligned}\frac{\alpha}{|\alpha|} =& u\prod_{j\in J}\frac{a_j+\epsilon_j ib_j}{|a_j+ib_j|}\\ =& u\left(1+O\left(\sum_{j\in J}|\arctan(b_j/a_j)|\right)\right) = u+O\left(\frac{1}{(\log\log x)^{1/11}}\right)\end{aligned}$$

where the unit u depends on α. Also for $\gamma \in \mathbb{Z}[i]$ with $\gamma\overline{\gamma} = r_0^{a_0}$, we have $\frac{\gamma}{|\gamma|} = u+O(1/(\log\log x)^{1/11})$ and for $\delta \in \mathbb{Z}[i]$ with $\delta\overline{\delta} = n$, we have $\frac{\delta}{|\delta|} = u + O(\varepsilon)$. Hence by this and (5.12)

$$\begin{aligned}\sum_{\substack{\alpha\in\mathbb{Z}[i]\\ \alpha\overline{\alpha}=Q_0'}}\sum_{\substack{\beta\in\mathbb{Z}[i]\\ \beta\overline{\beta}=m_0}}\sum_{\substack{\gamma\in\mathbb{Z}[i]\\ \gamma\overline{\gamma}=r_0^{a_0}}}\sum_{\substack{\delta\in\mathbb{Z}[i]\\ \delta\overline{\delta}=n}} f\left(\frac{\alpha\beta\gamma}{|\alpha\beta\gamma|}\right) =& \sum_{\substack{\alpha\in\mathbb{Z}[i]\\ \alpha\overline{\alpha}=Q_0'}}\sum_{\substack{\gamma\in\mathbb{Z}[i]\\ \gamma\overline{\gamma}=r_0^{a_0}}}\sum_{\substack{\delta\in\mathbb{Z}[i]\\ \delta\overline{\delta}=n}}\left(\sum_{\substack{\beta\in\mathbb{Z}[i]\\ \beta\overline{\beta}=m_0}} f\left(\frac{u_{\alpha,\gamma,\delta}\cdot\beta}{|\beta|}\right)\right) + O(\varepsilon r(m))\\ =& r(Q_0)r(r_0^{a_0})r(n)\sum_{a^2+b^2=m_0} f\left(\frac{a+ib}{|a+ib|}\right) + O(\varepsilon r(m)),\end{aligned}$$

thereby proving (5.10).

The proof of (5.11) follows along the same lines upon noting that for $m = Q_0 n \in \mathcal{M}_2(x)$ we can write $m^+ = Q_1' r_1^{a_1}\frac{m_1}{(m_1,m_0)}g'\ell_n$ where ℓ_n is a sum of two squares. Note that $Q_1', \frac{m_1}{(m_1,m_0)}, r_1^{a_1}, g', \ell_n$ are pairwise co-prime by construction since all the prime divisors of ℓ_n are $\ge y$; the latter also implies that $\#\{p|\ell_n\} \le 2/\eta$. $\square$

5.3. **Proof of Theorem 1.1.** WLOG we can assume all the prime factors of m_0 are congruent to 1 (mod 4) (see (5.13)). Let Q_0, Q_1 be as in (5.6) and $\mathcal{M}_1(x)$ be as in (5.7) and recall for $m \in \mathcal{M}_1(x)$ that $m = Q_0 n$ where $n \in \mathcal{N}_1(x)$ and $\mathcal{N}_1$ is as in (4.1). By (4.8) and Lemma 4.4 it follows that for all but at most $o(\#\mathcal{M}_1(x))$ integers $m \in \mathcal{M}_1(x)$ that $m^+ = m+4$, $(\log x)^{1/4-\nu} \le r(m^+) \le (\log x)^{1/2+\nu}$ (for any fixed $\nu > 0$) and $4 \le r(m) \ll (\log x)^{o(1)}$. Combining this with Proposition 4.1 we get that for all but $o(\#\mathcal{M}_1(x))$ integers $m \in \mathcal{M}_1(x)$ that $\lambda_m - m = o(1)$ and moreover

$$\lambda_m - m \asymp \begin{cases}\dfrac{r(m)}{\log\lambda_m} & \text{in the weak coupling quantization,}\\ \dfrac{r(m)}{r(m^+)} & \text{in the strong coupling quantization.}\end{cases} \tag{5.14}$$

Also note that for such m as above, we also have $|\lambda_m - m^+| \ge 3$. Hence, using the above estimate along with (4.8) and (4.10) with $U = (\log\log x)^5$ we get for all but at most $o(\#\mathcal{M}_1(x))$ integers $m \in \mathcal{M}_1(x)$ that (in both cases)

$$\begin{aligned}\sum_{\ell\ge 0}\frac{r(\ell)}{(\ell-\lambda_m)^2} =& \frac{r(m)}{(m-\lambda_m)^2} + \frac{r(m^+)}{(m^+-\lambda_m)^2} + o(1)\\ =& \frac{r(m)}{(m-\lambda_m)^2}\left(1+O\left(\frac{r(m^+)(m-\lambda_m)^2}{r(m)}\right)\right) + o(1)\\ =& \frac{r(m)}{(m-\lambda_m)^2}\left(1+o(1)\right).\end{aligned} \tag{5.15}$$

Similarly, for all but at most $o(\#\mathcal{M}_1(x))$ integers $m \in \mathcal{M}_1(x)$

$$\sum_{\ell \geq 0} \frac{1}{(\ell - \lambda_m)^2} \sum_{a^2+b^2=\ell} f\left(\frac{a+ib}{|a+ib|}\right) = \frac{1}{(m-\lambda_m)^2} \sum_{a^2+b^2=m} f\left(\frac{a+ib}{|a+ib|}\right) + O(r(m^+)). \tag{5.16}$$

Therefore, combining (5.3), (5.14), (5.15) and (5.16) it follows for all but at most $o(\#\mathcal{M}_1(x))$ integers $m \in \mathcal{M}_1(x)$ we have that

$$\begin{aligned}
\langle \mathrm{Op}(f) g_{\lambda_m}, g_{\lambda_m} \rangle =& (1+o(1)) \frac{(m-\lambda_m)^2}{r(m)} \cdot \left(\frac{1}{(m-\lambda_m)^2} \sum_{a^2+b^2=m} f\left(\frac{a+ib}{|a+ib|}\right) + O(r(m^+)) \right) \\
=& (1+o(1)) \frac{1}{r(m)} \sum_{a^2+b^2=m} f\left(\frac{a+ib}{|a+ib|}\right) + o(1) \\
=& (1+o(1)) \frac{1}{r(m_0)} \sum_{a^2+b^2=m_0} f\left(\frac{a+ib}{|a+ib|}\right) + O(\varepsilon)
\end{aligned}$$

where the last step follows by (5.10). The estimate for the density of this subsequence of eigenvalues follows immediately from (5.8), noting that $Q_0, Q_1 \ll (\log x)^{o(1)}$.

5.4. **Proof of Theorem 1.2.** WLOG we can assume all the prime factors of m_0, m_1 are congruent to 1 (mod 4) (see 5.13). For sake of brevity let $\mathcal{L}_2 = \log\log x$. Let Q_0, Q_1 be as in (5.6) and $\mathcal{M}_2(x)$ be as in (5.7) and recall for $m \in \mathcal{M}_2(x)$ that $m = Q_0 n$ where $n \in \mathcal{N}_2(x)$ where $\mathcal{N}_2$ is as in (4.1). Note for each $m \in \mathcal{M}_2(x)$ that $r(m) \gg \mathcal{L}_2^{10}$. Also, by construction $r(m)/r(m+4) \asymp \frac{a_0+1}{a_1+1}$ where H, a_0, a_1 are also as in (5.6) and note $a_0, a_1 \leq \log \mathcal{L}_2$. Applying Proposition 4.1 we get that for all $m \in \mathcal{M}_2(x)$ outside an exceptional set of size $o(\#\mathcal{M}_2(x))$ that $m^+ = m+4$ and

$$\frac{\lambda_m - m}{m^+ - \lambda_m} = \frac{r(m)}{r(m^+)} \left(1 + O\left(\frac{\mathcal{L}_2^6}{r(m)}\right)\right) = \frac{r(m)}{r(m^+)} \left(1 + O\left(\mathcal{L}_2^{-4}\right)\right). \tag{5.17}$$

In particular, this implies that $\lambda_m - m \gg \mathcal{L}_2^{-1}$ and $m^+ - \lambda_m \gg \mathcal{L}_2^{-1}$. As before, using (4.8) and (4.10) with $U = \mathcal{L}_2^5$ we get for all but at most $o(\#\mathcal{M}_2(x))$ integers $m \in \mathcal{M}_2(x)$ that

$$\sum_{\ell \geq 0} \frac{r(\ell)}{(\ell - \lambda_m)^2} = \frac{r(m)}{(m-\lambda_m)^2} + \frac{r(m^+)}{(m^+ - \lambda_m)^2} + O(\mathcal{L}_2^{-1}) \tag{5.18}$$

and

$$\begin{aligned}
\sum_{\ell \geq 0} \frac{1}{(\ell - \lambda_m)^2} \sum_{a^2+b^2=\ell} f\left(\frac{a+ib}{|a+ib|}\right) =& \frac{1}{(m-\lambda_m)^2} \sum_{a^2+b^2=m} f\left(\frac{a+ib}{|a+ib|}\right) \\
&+ \frac{1}{(m^+ - \lambda_n)^2} \sum_{a^2+b^2=m^+} f\left(\frac{a+ib}{|a+ib|}\right) + O(\mathcal{L}_2^{-1}).
\end{aligned} \tag{5.19}$$

Let $C_m = \frac{1}{1+r(m)/r(m^+)}$. Applying (5.17),(5.18), and (5.19) in (5.3) we get

(5.20)

$$\begin{aligned}\langle \mathrm{Op}(f)g_{\lambda_m}, g_{\lambda_m}\rangle &= (1+O(\mathcal{L}_2^{-1}))\left(\frac{r(m)}{(m-\lambda_m)^2}+\frac{r(m^+)}{(m^+-\lambda_m)^2}\right)^{-1}\\ &\times\left(\frac{1}{(m-\lambda_m)^2}\sum_{a^2+b^2=m} f\left(\frac{a+ib}{|a+ib|}\right)+\frac{1}{(m^+-\lambda_m)^2}\sum_{a^2+b^2=m^+} f\left(\frac{a+ib}{|a+ib|}\right)+O(\mathcal{L}_2^{-1})\right)\\ &=\frac{C_m}{r(m)}\sum_{a^2+b^2=m} f\left(\frac{a+ib}{|a+ib|}\right)+\frac{1-C_m}{r(m^+)}\sum_{a^2+b^2=m^+} f\left(\frac{a+ib}{|a+ib|}\right)+O(\mathcal{L}_2^{-1}).\end{aligned}$$

Applying (5.10) to the first sum above we get

$$\frac{C_m}{r(m)}\sum_{a^2+b^2=m} f\left(\frac{a+ib}{|a+ib|}\right)=\frac{C_m}{r(m_0)}\sum_{a^2+b^2=m_0} f\left(\frac{a+ib}{|a+ib|}\right)+O(\varepsilon). \tag{5.21}$$

Similarly, applying (5.11) to the second sum on the RHS of (5.20) we get that

$$\frac{1-C_m}{r(m^+)}\sum_{a^2+b^2=m^+} f\left(\frac{a+ib}{|a+ib|}\right)=\frac{1-C_m}{r(m_1\ell_n)}\sum_{a^2+b^2=m_1\ell_n} f\left(\frac{a+ib}{|a+ib|}\right)+O(\mathcal{L}_2^{-1/11}), \tag{5.22}$$

for some integer ℓ_n with $\#\{p : p|\ell_m\} \le 2/\eta$ by (4.2). Using (5.21) and (5.22) in (5.20) completes the proof upon taking $\varepsilon = \mathcal{L}_2^{-1/2}$. The estimate for the density of this subsequence of eigenvalues follows from (5.9).

5.5. **Proof of Theorem 1.3.** The proof of Theorem 1.3 relies on the following hypothesis concerning the distribution of primes.

Hypothesis 1. *Let Q_1, Q_0 be as in (5.6) and $\varepsilon \ge (\log\log x)^{-1/2}$ be sufficiently small. Also let $y = x^\eta$ where $\eta > 0$ is sufficiently small. Then the number of solutions $(u, v) \in \mathbb{Z}^2$ to*

$$Q_1 u - Q_0 v = 4$$

where $v = p_1p_2$ and $u = p_3$ are primes satisfying $1_{\mathcal{P}_\varepsilon}(p_1)1_{\mathcal{P}'_\varepsilon}(p_2)1_{\mathcal{P}_\varepsilon}(p_3) = 1$, $p_3 > y$ such that $v \le x$ is

$$\gg \varepsilon^3 \frac{Q_0}{\varphi(Q_0Q_1)}\frac{x\log\log x}{(\log x)^2}.$$

where $\mathcal{P}_\varepsilon, \mathcal{P}'_\varepsilon$ are as in (2.1).

Proof of Theorem 1.3. Recall the definition of $\mathcal{N}_2$ given in (4.1). Let us define

$$\mathcal{N}_3 = \{n \in \mathcal{N}_2 : Q_0 n + 4 = Q_1 p, b(p) = 1, \ \&\ |\theta_p| \le \varepsilon\}.$$

Following (5.7) we also define

$$\mathcal{M}_3(x) = \{m \le x : m = Q_0 n \text{ and } n \in \mathcal{N}_3\}.$$

By Hypothesis 1 and (5.9) it follows that

$$\#\mathcal{M}_3(x) \asymp \varepsilon\#\mathcal{M}_2(x) \tag{5.23}$$

where we also have used an upper bound sieve to get that $\#\mathcal{M}_3(x) \ll \varepsilon\#\mathcal{M}_2(x)$. Observe that $\mathcal{M}_3(x) \subset \mathcal{M}_2(x)$ and the exceptional set in Proposition 4.1 is $o(\#\mathcal{M}_3(x))$ since we take $\varepsilon = (\log\log x)^{-1/4}$. Hence, we get that (5.17) holds for $m \in \mathcal{M}_3(x)$ outside an exceptional set of size $o(\#\mathcal{M}_3(x))$. Similarly, we can conclude that (5.18) and (5.19) also hold for all $m \in \mathcal{M}_3(x)$ outside

an exceptional set of size $o(\#\mathcal{M}_3(x))$. Therefore, arguing as in (5.20)–(5.22) we conclude that for $m \in \mathcal{M}_3(x)$ outside an exceptional set of size $o(\#\mathcal{M}_3(x))$ we have that
(5.24)
$$\langle \mathrm{Op}(f) g_{\lambda_m}, g_{\lambda_m} \rangle = \frac{C_m}{r(m_0)} \sum_{a^2+b^2=m_0} f\left(\frac{a+ib}{|a+ib|}\right) + \frac{1-C_m}{r(m_1\ell_n)} \sum_{a^2+b^2=m_1\ell_n} f\left(\frac{a+ib}{|a+ib|}\right) + O(\mathcal{L}_2^{-1/11})$$

where m_0, m_1 are arbitrary, fixed integers whose prime factors are all congruent to 1 (mod 4) and $C_m = 1/(1+r(m)/r(m+4))$. By our hypothesis we have that $\ell_n = p$ with $|\theta_p| \le \varepsilon$ and $(m_1, p) = 1$. Hence, repeating the argument used to prove (5.10) it follows that

$$\frac{1}{r(m_1\ell_n)} \sum_{a^2+b^2=m_1\ell_n} f\left(\frac{a+ib}{|a+ib|}\right) = \frac{1}{r(m_1)} \sum_{a^2+b^2=m_1} f\left(\frac{a+ib}{|a+ib|}\right) + O(\varepsilon). \tag{5.25}$$

Given $0 < c < 1$ with $c = d/e \in \mathbb{Q}$ we will now specify our choice of a_0, a_1 (from (5.5)). Recall we allow a_0, a_1 to grow slowly with x and Q_0', Q_1' have the same number of prime factors. Also, by construction $r(\frac{m_1}{(m_0,m_1)} g') = r(m_1)$. Let $\mathcal{L} = \lfloor (\log\log\log x)^{1/2} \rfloor$. We take

$$a_0 = 2(e-d) r(m_1) \mathcal{L} \qquad \text{and} \qquad a_1 = d r(m_0) \mathcal{L}.$$

Hence,

$$C_m = \frac{1}{1 + \frac{2r(m_0)(a_0+1)}{4r(m_1)(a_1+1)}} = \frac{d}{e} + o(1). \tag{5.26}$$

We are now ready to complete the proof. Given any attainable measures μ_{∞_0}, μ_{∞_1} and $0 \le c \le 1$ we can take $\{m_{0,j}\}_j$ $\{m_{1,j}\}$ such that $\mu_{0,j}$ weakly converges to μ_{∞_0} and $\mu_{1,j}$ weakly converges to μ_{∞_1}, as $j \to \infty$. We also take $\{a_{0,j}\}_j, \{a_{1,j}\}_j$ so that $d_j/e_j \to c$ as $j \to \infty$. Therefore, by (5.24),(5.25), and (5.26) we conclude that there exists $\{\lambda_\ell\}_\ell \subset \Lambda_{\mathrm{new}}$ such that

$$\langle \mathrm{Op}(f) g_{\lambda_\ell}, g_{\lambda_\ell} \rangle \xrightarrow{\ell \to \infty} c \int_{S^1} f \, d\mu_{\infty_0} + (1-c) \int_{S^1} f \, d\mu_{\infty_1}.$$

□

Appendix A. Arithmetic over $\mathbb{Q}(i)$

Consider the number field $\mathbb{Q}(i)$ with ring of integers $\mathbb{Z}[i]$. For $\mathfrak{b}$ a non-zero integral ideal of $\mathbb{Z}[i]$ the residue classes $\alpha \pmod{\mathfrak{b}}$, where (α) and $\mathfrak{b}$ are relatively prime ideals, form the multiplicative group $(\mathbb{Z}[i]/\mathfrak{b})^*$. We now summarize some well-known facts, which may be found in [35] or [19]. A *Dirichlet character* $\pmod{\mathfrak{b}}$ is a group homomorphism

$$\chi : (\mathbb{Z}[i]/\mathfrak{b})^* \to S^1.$$

We extend χ to all of $\mathbb{Z}[i]$ by setting $\chi(\mathfrak{a}) = 0$ for $\mathfrak{a}$ and $\mathfrak{b}$ which are not relatively prime. Let I denote multiplicative group of non-zero fractional ideals and $I_{\mathfrak{b}} = \{\mathfrak{a} \in I : \mathfrak{a} \text{ and } \mathfrak{b} \text{ are relatively prime}\}$. A *Hecke Großencharakter* $\pmod{\mathfrak{b}}$ is a homomorphism $\psi : I_{\mathfrak{b}} \to \mathbb{C} \setminus \{0\}$ for which there exists a pair of homomorphisms

$$\chi : (\mathbb{Z}[i]/\mathfrak{b})^* \to S^1, \qquad \chi_\infty : \mathbb{C}^* \to S^1$$

such that for an ideal (α) with $\alpha \in \mathbb{Z}[i]$

$$\psi((\alpha)) = \chi(\alpha) \chi_\infty(\alpha).$$

Conversely, given any $\chi \pmod{\mathfrak{b}}$ and χ_∞ there exists a Großencharakter $\psi \pmod{\mathfrak{b}}$ such that $\psi = \chi \cdot \chi_\infty$ provided that $\chi(u)\chi_\infty(u) = 1$ for each unit $u \in \mathbb{Z}[i]$.

In particular, for $4|k$ and $\mathfrak{a} = (\alpha)$ a non-negative integer

$$\psi(\mathfrak{a}) = \left(\frac{\alpha}{|\alpha|}\right)^k$$

is a Hecke Großenchakter $\pmod 1$ and these Hecke Großencharakteren can be used to detect primes in sectors. Additionally, given a positive rational integer q with $(4, q) = 1$ the homomorphism

$$\chi \circ N : I_q \to S^1$$

given by $(\chi \circ N)(\mathfrak{a}) = \chi(N(\mathfrak{a}))$ is a Dirichlet character $\pmod q$, where χ is a Dirichlet character $\pmod q$ for $\mathbb{Z}$, that is $\chi : (\mathbb{Z}/(q))^* \to S^1$, where $N\mathfrak{a}$ is the norm of $\mathfrak{a}$. Hence, for $4|k$

$$\psi(\mathfrak{a}) = (\chi \circ N)(\alpha) \left(\frac{\alpha}{|\alpha|}\right)^k$$

is a Hecke Großencharakter with modulus q and frequency k, where $\mathfrak{a} = (\alpha)$. (A priori α is only defined up to multiplication by i, but for these characters the choice does not matter). The L-function attached to the Großencharakter ψ given by

$$L(s, \psi) = \sum_{\mathfrak{a}} \frac{\psi(\mathfrak{a})}{N(\mathfrak{a})^s},$$

has a functional equation and admits an analytic continuation to $\mathbb{C} \setminus \{1\}$.

Moreover, if ψ is not a real character, $L(s, \psi)$ has a standard zero free region. That is, we have

$$L(\sigma + it, \psi) \neq 0 \qquad \text{for} \qquad \sigma > 1 - \frac{c}{\log(q(|t| + 1)(|k| + 1))}$$

(see [19, Section 5.10]). In particular, if $k \neq 0$,

$$\sum_{N(\pi) \le x} \chi(N(\pi)) \left(\frac{\pi}{|\pi|}\right)^k \ll ((|k| + 1)q) \cdot x \exp\left(-c\sqrt{\log x}\right),$$

where the summation is over prime ideal $\mathfrak{p} = (\pi)$ with norm $\le x$.

Furthermore, for $k = 0$ the same estimate holds for any complex $\chi \pmod q$. However for $k = 0$ and $\chi \pmod q$ a real character, there may be a possible Siegel zero and in this case we have Siegel's estimate (see Section 5.9 of [19])

$$L(\sigma + it, \chi) \neq 0 \qquad \text{for} \qquad \sigma \ge 1 - \frac{c(\epsilon)}{q^\epsilon}$$

for any $\epsilon > 0$. Consequently, we have the Siegel-Walfisz type prime number theorem for $(a, q) = 1$ and $(q, 2) = 1$

$$\sum_{\substack{N(\pi) \le x \\ N(\pi) \equiv a \pmod q \\ 0 \le \arg \pi \le \varepsilon}} 1 = \frac{1}{\varphi(q)} \sum_{\substack{N(\pi) \le x \\ (N(\pi), q) = 1 \\ 0 \le \arg \pi \le \varepsilon}} 1 + O\left(\frac{x}{(\log x)^A}\right) \tag{A.1}$$

for any $A \ge 1$. (After multiplication by i^l for some l we can ensure that $\theta = \arg i^l \pi \in [0, \pi/2)$; we will let $\arg \pi$ denote this angle.)

Recall that a prime $p \equiv 3 \pmod 4$ is inert in $\mathbb{Z}[i]$; additionally, a prime $p \equiv 1 \pmod 4$ splits in $\mathbb{Z}[i]$ so that $p = \pi\overline{\pi} = a^2 + b^2$, where π is a prime in $\mathbb{Z}[i]$. Writing

$$\mathcal{B}(x; q, a, \varepsilon) = \sum_{\substack{n \le x \\ n \equiv a \pmod q}} (1_{\mathcal{P}_\varepsilon} * 1_{\mathcal{P}'_\varepsilon})(n) - \frac{1}{\varphi(q)} \sum_{\substack{n \le x \\ (n, q) = 1}} (1_{\mathcal{P}_\varepsilon} * 1_{\mathcal{P}'_\varepsilon})(n),$$

formula (A.1) gives, for $(a,q)=1$ and $(q,2)=1$, that

$$|\mathcal{B}(x;q,a,\varepsilon)| \ll \frac{x}{(\log x)^A}, \tag{A.2}$$

for $q \le (\log x)^A$. In addition it is worth noting that (A.1) also implies

$$\sum_{\substack{n\le x\\ n\equiv a \pmod q}} (1_{\mathcal{P}_\varepsilon} * 1_{\mathcal{P}'_\varepsilon})(n) \sim \frac{4\varepsilon^2}{\varphi(q)} \frac{x\log\log x}{\log x}. \tag{A.3}$$

We are now ready to state the following result which is an analog of the Bombieri-Vinogradov Theorem.

Theorem A.1. *There exists B_0 sufficiently large so that*

$$\sum_{\substack{q\le Q\\ (q,2)=1}} \max_{(a,q)=1} |\mathcal{B}(x;q,a,\varepsilon)| \ll \frac{x}{(\log x)^{10}}$$

for $Q \le x^{1/2}/(\log x)^{B_0}$.

Let $\mathcal{S} \subset \mathbb{N}$. A sequence of complex numbers $\{\beta_n\}$ with $|\beta_n| \le \tau(n)$ satisfies the *Siegel-Walfisz property for $\mathcal{S}$* provided that for every $q \in \mathcal{S}$ and $A \ge 0$ and $N \ge 2$ we have

$$\sum_{\substack{n\le N\\ n\equiv a \pmod q}} \beta_n = \frac{1}{\varphi(q)} \sum_{\substack{n\le N\\ (n,q)=1}} \beta_n + O\left(\frac{N}{(\log N)^A}\right)$$

for every $a \in \mathbb{Z}$ with $(a,q)=1$.

A.1. **An application of the large sieve.** We next recall a consequence of the large sieve, which follows applying a minor modification of Theorem 9.17 of [15].

Lemma A.1. *Let $A \ge 1$ and $Q = x^{1/2}(\log x)^{-B}$ where $B = B(A)$ is sufficiently large. Suppose $\{\beta_n\}$ satisfies the Siegel-Walfisz property for all q with $(q,2)=1$. Then for any sequence $\{\alpha_n\}$ of complex numbers such that $|\alpha_n| \le \tau(n)$*

$$\sum_{\substack{q\le Q\\ (q,2)=1}} \max_{(a,q)=1} \left| \sum_{\substack{mn\le x\\ m,n\le \frac{x}{(\log x)^B}\\ mn\equiv a \pmod q}} \beta_m\alpha_n - \frac{1}{\varphi(q)} \sum_{\substack{mn\le x\\ m,n\le \frac{x}{(\log x)^B}\\ (mn,q)=1}} \beta_m\alpha_n \right| \ll \frac{x}{(\log x)^A}.$$

Proof of Theorem A.1. By (A.2) the sequence $\beta_n = 1_{\mathcal{P}_\varepsilon}(n)$ satisfies the Siegel-Walfisz condition for all q with $(q,2)=1$. Take $\alpha_n = 1_{\mathcal{P}'_\varepsilon}(n)$ and note that (cf. (2.1))

$$\sum_{\substack{n\le x\\ n\equiv a \pmod q}} (1_{\mathcal{P}_\varepsilon} * 1_{\mathcal{P}'_\varepsilon})(n) = \sum_{\substack{mn\le x\\ m,n\le \frac{x}{(\log x)^{B_0}}\\ mn\equiv a \pmod q}} 1_{\mathcal{P}_\varepsilon}(m)1_{\mathcal{P}'_\varepsilon}(n)$$

and

$$\sum_{\substack{n\le x\\ (n,q)=1}} (1_{\mathcal{P}_\varepsilon} * 1_{\mathcal{P}'_\varepsilon})(n) = \sum_{\substack{mn\le x\\ m,n\le \frac{x}{(\log x)^{B_0}}\\ (mn,q)=1}} 1_{\mathcal{P}_\varepsilon}(m)1_{\mathcal{P}'_\varepsilon}(n).$$

Hence, applying Lemma A.1 completes the proof. $\square$

A.2. **Gaussian integers in sectors with norms in progressions.** The goal of this section is to show that a result of Smith [43] (also cf. [46]) holds for Gaussian integers in sectors. We recall that for $\alpha \in \mathbb{Z}[i]$, $N(\alpha) = |\alpha|^2$ denotes the norm of α. For $a, q > 0$ define

$$\eta_a(q) := |\{\alpha_1, \alpha_2 \pmod q : \alpha_1^2 + \alpha_2^2 \equiv a \pmod q\}|.$$

Proposition A.1. *Let $a, q > 0$ be integers and put $g = (a, q)$. Given an angle θ and $\epsilon \in (0, 2\pi)$, let $S = S_{\epsilon,\theta}$ denote the set of lattice points $\alpha \in \mathbb{Z}[i]$ contained in the sector defined by*[3] $|\arg(\alpha) - \theta| < \epsilon/2$*. Then, uniformly for $\epsilon > 0$,*

$$|\{\alpha \in S : N(\alpha) \equiv a \pmod q, N(\alpha) \le x\}|$$
$$= \frac{\epsilon x \eta_a(q)}{q^2} + O\left(\frac{x^{1-\delta/3}}{q}\right)$$

provided that $q^3 g < x^{2(1-2\delta)}$ for $\delta > 0$.

We begin by showing that solutions to $\alpha_1^2 + \alpha_2^2 \equiv a \pmod q$ is well distributed in fairly small boxes. Given q, let $f : (\mathbb{Z}/q\mathbb{Z})^2 \to \mathbb{C}$ denote the characteristic function of the set $\{(\alpha_1, \alpha_2) \in (\mathbb{Z}/q\mathbb{Z})^2 : \alpha_1^2 + \alpha_2^2 \equiv a \pmod q\}$. With the modulo q Fourier transform given by

$$\widehat{f}(\xi_1, \xi_2) := \sum_{\alpha_1, \alpha_2 \pmod q} f(\alpha_1, \alpha_2) e^{-2\pi i (\xi_1 \alpha_1 + \xi_2 \alpha_2)/q} \tag{A.4}$$

we recall the following estimate by Tolev [46]:

$$|\widehat{f}(\xi_1, \xi_2)| \ll q^{1/2} \tau(q)^2 (q, \xi_1, \xi_2)^{1/2} (q, a, \xi_1^2 + \xi_2^2)^{1/2} \le q^{1/2} \tau(q)^2 (q, \xi_1, \xi_2)^{1/2} (q, a)^{1/2} \tag{A.5}$$

By the Chinese remainder theorem, $\eta_a(q)$ is multiplicative in q, and we note that $\widehat{f}(0,0) = \eta_a(q)$.

Let $B \subset [0, q) \times [0, q)$ be a "box" with side lengths T, and let $g = g_B$ denote the characteristic function of $B \cap (\mathbb{Z}/q\mathbb{Z})^2$. By standard estimates (from summing a geometric series) we have, for $\xi_1, \xi_2 \neq 0$,

$$\widehat{g}(\xi_1, \xi_2) \ll q^2 / |\xi_1 \xi_2|, \tag{A.6}$$

for $\xi_1 \neq 0$,

$$\widehat{g}(\xi_1, 0) \ll Tq / |\xi_1|, \tag{A.7}$$

(and similarly for $\xi_2 \neq 0$), and trivially

$$\widehat{g}(0, 0) = T^2.$$

Lemma A.2. *Let $g = (a, q)$. Then*

$$|\{(\alpha_1, \alpha_2) \in B : \alpha_1^2 + \alpha_2^2 \equiv a \pmod q\}| = T^2 \cdot \frac{\eta_a(q)}{q^2} + O(q^{1/2} \tau(q)^3 \log(q)^2 g^{1/2})$$

Proof. By Fourier analysis on $(\mathbb{Z}/q\mathbb{Z})^2$ (i.e., Plancherel's theorem for finite abelian groups) we have

$$|\{(\alpha_1, \alpha_2) \in B : \alpha_1^2 + \alpha_2^2 \equiv a \pmod q\}| = \sum_{\alpha_1, \alpha_2 \pmod q} f(\alpha_1, \alpha_2) g(\alpha_1, \alpha_2)$$

$$= \frac{1}{q^2} \sum_{\xi_1, \xi_2 \pmod q} \widehat{f}(\xi_1, \xi_2) \overline{\widehat{g}(\xi_1, \xi_2)}$$

[3] By $\arg(\alpha)$ we denote the complex argument chosen in such a way that it is single valued in an $\epsilon/2$-neighborhood of θ.

The main term is given by $\xi_1 = \xi_2 = 0$ and equals

$$\frac{\widehat{f}(0,0)\widehat{g}(0,0)}{q^2} = T^2 \frac{\eta_a(q)}{q^2}$$

Using (A.5) and (A.7) the contribution from (say) $\xi_1 = 0$ and $\xi_2 \neq 0$ is

$$\text{(A.8)} \qquad \ll \frac{1}{q^2} \sum_{\xi_2=1}^{q-1} \frac{Tq}{\xi_2} q^{1/2} \tau(q)^2 (q, \xi_2)^{1/2} g^{1/2} \ll \frac{Tq^{3/2}\tau(q)^2 g^{1/2}}{q^2} \sum_{d|q} \sum_{0<\xi_2<q/d} \frac{d^{1/2}}{d\xi_2}$$

$$\ll \frac{T\tau(q)^3 \log(q) g^{1/2}}{q^{1/2}} = O(q^{1/2}\tau(q)^3 \log(q) g^{1/2}).$$

The contribution from terms $\xi_2 = 0$ and $\xi_1 \neq 0$ is bounded similarly.

As for the terms $\xi_1, \xi_2 \neq 0$, we have by (A.5)

$$\frac{1}{q^2} \sum_{\xi_1,\xi_2 \neq 0} \widehat{f}(\xi_1,\xi_2)\overline{\widehat{g}(\xi_1,\xi_2)} \ll \frac{q^{1/2}\tau(q)^2}{q^2} \sum_{\xi_1,\xi_2\neq 0} \frac{q^2}{\xi_1\xi_2}(q,\xi_1,\xi_2)^{1/2} g^{1/2}$$

$$= q^{1/2}\tau(q)^2 \sum_{d|q} \sum_{0<\xi_1,\xi_2 \leq q/d} \frac{d^{1/2}g^{1/2}}{d^2\xi_1\xi_2} \ll q^{1/2}\tau(q)^2 \log(q)^2 g^{1/2}.$$

□

Concluding the proof of Proposition A.1. Take $T = x^{(1-\delta)/2}$. The case $T > q$ is straightforward using a simple tiling argument, and we only give details for $T \leq q$.

By a simple geometry of numbers argument, we may "tile" the sector S, intersected with a ball of radius $x^{1/2}$, with $\epsilon x/T^2 + O(x^{1/2}/T)$ boxes B (with side lengths T) entirely contained in the sector, and with $O(x^{1/2}/T)$ boxes intersecting the boundary. By Lemma A.2, each box B contains

$$T^2 \cdot \frac{\eta_a(q)}{q^2} + O(q^{1/2}\tau(q)^2 \log(q)^2 g^{1/2})$$

points satisfying $\alpha_1^2 + \alpha_2^2 \equiv a \pmod{q}$.

As $\eta_a(q) < q^{1+o(1)}$ (cf. [5, Lemma 2.8]), we find that the number of lattice points in the sector is

$$(\epsilon x/T^2 + O(x^{1/2}/T))(T^2 \cdot \frac{\eta_a(q)}{q^2} + O(q^{1/2}\tau(q)^3 \log(q)^2 g^{1/2}))$$
$$= \frac{\epsilon \eta_a(q) x}{q^2} + O\left(\frac{x^{1-\delta/2}}{q^{1-o(1)}} + \epsilon g^{1/2} q^{1/2+o(1)} x^{\delta}\right).$$

For $q^3 g < x^{2(1-2\delta)}$ the error term is $\ll \frac{x^{1-\delta/3}}{q}$. □

A.3. **Proof of Lemma 2.3.** We may assume $(Q, q) = 1$ otherwise the result is trivial. Let $\delta > 0$ be sufficiently small but fixed and set

$$r_\varepsilon(n) = \sum_{\substack{a^2+b^2=n \\ |\arg(a+ib)| \leq \varepsilon}} 1.$$

Also, for $n \in \mathbb{N}$ and $z > 0$ let $\widetilde{P}_n(z) = \prod_{2<p<z} p$. Let $\Lambda_1 = \{\lambda_d\}$, $\Lambda' = \{\lambda'_e\}$ be upper bound sieves of level $D = x^\delta$ with $(d, 2q) = 1$ and $(e, 2Q) = 1$. Then for $z = x^{\delta/2}$ we have

$$\sum_{\substack{p=a^2+b^2\leq x \\ |\arg(a+ib)|\leq\varepsilon \\ qp+4=Qp_1 \text{where } p_1 \text{ is prime}}} 1 \leq \sum_{m\leq qx+4} \sum_{\substack{n\leq x \\ qn+4=Qm \\ (m,\widetilde{P}_q(z))=1 \\ (n,\widetilde{P}_Q(z))=1}} r_\varepsilon(n) + O(x^{\delta/2})$$
$$\leq \sum_{m\leq qx+4} \sum_{\substack{n\leq x \\ qn+4=Qm}} r_\varepsilon(n)(\lambda' * 1)\,(n)\,(\lambda * 1)\,(m) + O(x^{\delta/2}).$$

Switching order of summation we have that the sum on the LHS above is

$$\begin{aligned} &= \sum_{\substack{d,e<D \\ (d,e)=1 \\ (d,2q)=1,(e,2Q)=1}} \lambda_d\lambda'_e \sum_{\substack{n\leq x \\ e|n}} r_\varepsilon(n) \sum_{\substack{m\leq qx+4 \\ d|m \\ qn+4=Qm}} 1 \\ &= \sum_{\substack{d,e<D \\ (d,e)=1 \\ (d,2q)=1,(e,2Q)=1}} \lambda_d\lambda'_e \sum_{\substack{n\leq x \\ n\equiv\gamma \pmod{Qed}}} r_\varepsilon(n) \end{aligned} \tag{A.9}$$

since the inner sum in the first equation above consists of precisely one term provided that $qn+4 \equiv 0 \pmod{Qd}$ and is empty otherwise. Also, here $\gamma = -4e\overline{e}\overline{q}$ where $q\overline{q} \equiv 1 \pmod{Qd}$ and $e\overline{e} \equiv 1 \pmod{Qd}$. In particular, $(\gamma, Qed) = e$.

Let us note some properties of the function $\eta_a(q)$. Recall, $\eta_a(\cdot)$ is multiplicative. Moreover, for $p > 2$ and $\ell \geq 1$

$$\eta_a(p^\ell) = p^\ell \sum_{0\leq j\leq \ell} \frac{\chi_4(p)^j}{p^j} c_{p^j}(a) \tag{A.10}$$

and for any $a, q \geq 1$

$$\eta_q(q) \ll \frac{q^2}{\varphi(q)}\tau((a,q)) \tag{A.11}$$

(see [5, Eqn. (2.20) and Lemma 2.8]) where

$$c_q(a) = \sum_{\substack{b \pmod{q} \\ (b,q)=1}} e\left(\frac{ab}{q}\right) = \frac{\varphi(q)}{\varphi(q/(q,a))}\mu(q/(q,a)) \tag{A.12}$$

is the Ramanujan sum and χ_4 is the non-principal Dirichlet character $\pmod 4$. In particular note that if $(a, q) = g$ then $\eta_a(q) = \eta_g(q)$ for odd q.

By Proposition A.1, (A.10), (A.11) and recalling that $(Qed, \gamma) = e$ we get the RHS of (A.9) equals

$$2\varepsilon x \sum_{\substack{d,e<D \\ (d,e)=1 \\ (d,2q)=1,(e,2Q)=1}} \frac{\lambda_d\lambda'_e}{(Qed)^2}\eta_\gamma(Qed) + O\left(\frac{x^{1-\delta/4}}{Q}\right)$$
$$= \frac{2\varepsilon x\eta_1(Q)}{Q^2} \sum_{\substack{d,e<D \\ (d,e)=1 \\ (d,2q)=1,(e,2Q)=1}} \frac{\lambda_d\lambda'_e}{(ed)^2}\frac{\eta_1(Qd)\eta_e(e)}{\eta_1(Q)} + O\left(\frac{x^{1-\delta/4}}{Q}\right)$$

provided that $Q^3D^7 < x^{2(1-2\delta)}$ which we rewrite as $Q < x^{2/3-11\delta/3}$. Using Theorem 2.3 in the form of (2.8), and noting that $\eta_1(Qd)/\eta_1(Q)$ is a multiplicative function, we get that the above sum is

$$\ll \frac{\varepsilon x \eta_1(Q)}{Q^2} \prod_{\substack{p<D \\ (p,2q)=1}} \left(1 - \frac{\eta_1(Qp)}{p^2\eta_1(Q)}\right) \prod_{\substack{p<D \\ (p,2Q)=1}} \left(1 - \frac{\eta_p(p)}{p^2}\right). \tag{A.13}$$

To evaluate the Euler products we use (A.10) to get $\eta_p(p) = p(1+\chi_4(p)-\frac{1}{p})$, $\eta_1(Qp)/\eta_1(Q) = p + O(1)$ and $\eta_1(Q) = Q\prod_{p|Q}\left(1-\frac{\chi_4(p)}{p}\right)$. Hence, by these estimates we get that (A.13) is

$$\begin{aligned}
&\ll \frac{\varepsilon x \eta_1(Q)}{Q^2} \prod_{p|Q}\left(1+\frac{\chi_4(p)+1}{p}\right)\prod_{p|q}\left(1+\frac{1}{p}\right)\cdot\frac{1}{(\log D)^2} \\
&\ll \frac{q}{\varphi(q)}\cdot\frac{\varepsilon x}{Q\delta^2(\log x)^2}\prod_{p|Q}\left(1+\frac{1}{p}\right) \ll \frac{q}{\varphi(q)}\cdot\frac{\varepsilon x}{\varphi(Q)\delta^2(\log x)^2}
\end{aligned}$$

for $Q < x^{2/3-11\delta/3}$ which completes the proof, since $\delta > 0$ is arbitrary.

Appendix B. Non-attainable quantum limits

Given an integer n such that $r(n) > 0$, define a probability measure μ_n on the unit circle by

$$\mu_n := \frac{1}{r(n)} \sum_{\lambda\in\mathbb{Z}[i]:|\lambda|^2=n} \delta_{\lambda/|\lambda|},$$

i.e., μ_n is obtained by projection the set of $\mathbb{Z}^2$-lattice points on a circle of radius $n^{1/2}$ to the unit circle and δ here denotes the Dirac delta function. A measure μ is said to be *attainable* if μ is a weak* limit of some subsequence of measures μ_{n_i}. A partial classification of the set of attainable measures were given in [30] in terms of their Fourier coefficients. Namely, for $k \in \mathbb{Z}$, let $\widehat{\mu}(k) := \int z^k\, d\mu(z)$ denote the k-th Fourier coefficent of μ. By [30, Theorem 1.3], the inequalities

$$2\widehat{\mu}(4)^2 - 1 \le \widehat{\mu}(8) \le \max(\widehat{\mu}(4)^4, (2|\widehat{\mu}(4)|-1)^2)$$

holds if μ is attainable. In particular, for $\gamma > 0$ small and $\widehat{\mu}(4) = 1-\gamma$, we must have $\widehat{\mu}(8) = 1 - 4\gamma + O(\gamma^2)$.

Now, by Theorem 1.2, there exists quantum limits that are convex combinations $c\nu_1 + (1-c)\nu_2$ for $c > 0$ arbitrary small, and where ν_1 is the uniform measure (with $(\widehat{\nu_2}(4), \widehat{\nu_2}(8)) = (0,0)$), and ν_2 is a Cilleruello type measure, i.e., localized on the four points $\pm 1, \pm i$, and with $(\widehat{\nu_2}(4), \widehat{\nu_2}(8)) = (1,1)$. Clearly such convex combinations cannot be attainable for c small.

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Department of Mathematics, KTH Royal Institute of Technology, SE-100 44 Stockholm, Sweden
E-mail address: kurlberg@kth.se

School of Mathematical Sciences, Queen Mary University, E1 4NS London, UK
E-mail address: s.lester@qmul.ac.uk

Mathematics Unit, Afeka Tel Aviv Academic College of Engineering, Mivtza Kadesh St., Tel-Aviv, Israel
E-mail address: LiorR@afeka.ac.il